variance:-

variance is approx. as.

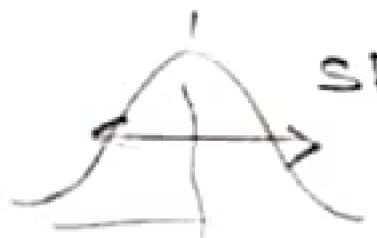
MIND THE "N-1".

$$\boxed{\text{variance} = s^2 = \frac{1}{N-1}\sum_{1}^{N}(\bar{x}-x_i)^2}$$

N-1 : very technical reason.

computed for Large N values.

$\sqrt{\text{variance}} = \sigma = S =$ standard deviation.



* $x_i \longrightarrow ax_i+b$

Standard division $\times a$

practical application:-

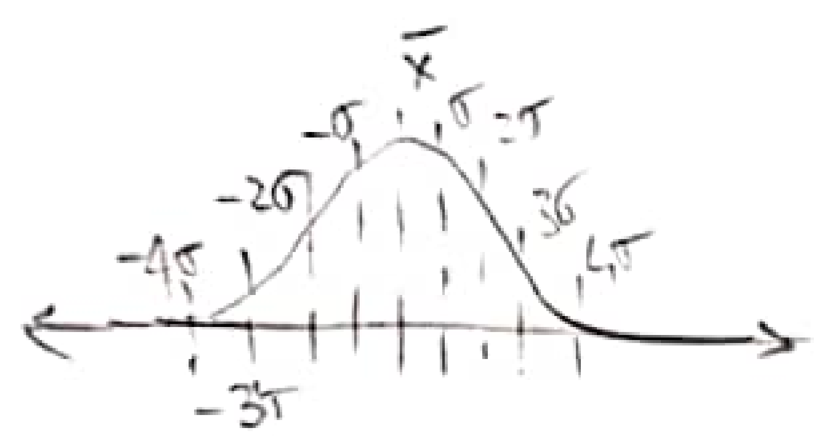
1) 100gm chips is sold. make sure avg ≈ 100gm & σ is low.

2) definitⁿ of medical dieases:- "deficiency of RBC" "osteoporosis". low bone density. ⋮ we have limits na!.

"Two sided -

→ chebyshev's inequality:- suggests upper limit.

"The proportion of sample points; K or more than K (K>0) standard deviations away from the sample mean, is less than or equal to $\frac{1}{K^2}$" (proportion)

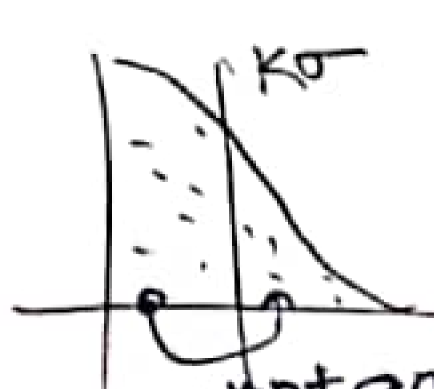


*Bcoz; if they are more than $\frac{N}{K^2}$; their squared sum contribution in σ^2 equⁿ would overshoot σ^2 itself.

Bounds are LOOSE, but NOT wrong.

$$\boxed{S_K = \{x_i : |x_i-\bar{x}| \geq K\sigma\} \quad \text{then} \quad \frac{|S_K|}{N} \leq \frac{1}{K^2}}$$

* Gives lower bound on elements within $K\sigma$ & upper bound on those, out of $K\sigma$.



proof:

$$\sigma^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$$

$$(n-1)\sigma^2 = \underbrace{\sum_{x_i \notin S_K}(x_i-\bar{x})^2}_{\text{near to mean}} + \underbrace{\sum_{x_i \in S_K}(x_i-\bar{x})^2}_{\text{far to mean.}}$$

$$\therefore (n-1)\sigma^2 > \sum_{x_i \in S_K}\underbrace{(x_i-x_i)^2}_{|\,| \geq SK\sigma.}$$

$$\therefore (n-1)\sigma^2 \geq K^2\sigma^2|S_K|$$

$$\Leftarrow \frac{1}{K^2} \geq \frac{|S_K|}{N}$$

$$\frac{|S_K|}{N} \leq \frac{1}{K^2}$$

101.

→ One sided chebysev's inequality: is stronger

Chebyshev- cantelli inequality:-

"the proportion of sample points, K are more than K s.d.s away from sample mean, & greater than the sample mean,

is less than or equal to $\frac{1}{1+K^2}$"

$$\boxed{S_K = \{x : x_i - \bar{x} \geq K\sigma\} \qquad \frac{|S_K|}{N} \leq \frac{1}{1+K^2}}$$

Not an absolute value.

$\boxed{\text{Stronger than 2-side chebysev.}}$

but weaker, when we use this, to compute:

$S_K = \{x : |x_i - \bar{x}| \geq K\sigma\}$

$\frac{|S_K|}{N} \leq \underbrace{\frac{2}{1+K^2}}$ weaker than $\frac{1}{K^2}$

Proof:-

$\{x_i\}_{i=1}^{N}$.

let $y_i = x_i - \bar{x}$

$$\sum_{i=1}^{n} (y_i + b)^2 \geq \sum_{i: y_i \geq K\sigma} (y_i + b)^2 \qquad b > 0$$

$$\geq \sum_{y_i \geq K\sigma} (K\sigma + b)^2$$

$$\sum_{i=1}^{n} y_i^2 + nb^2 \geq \quad = |S_K| \cdot (K\sigma + b)^2$$

$$(n-1)\cdot\sigma^2 + nb^2 \geq |S_K| \cdot (K\sigma + b)^2$$

$$\therefore \quad |S_K| \leq \frac{(n-1)\sigma^2 + nb^2}{(K\sigma + b)^2}$$

$$\frac{|S_K|}{n} < \frac{\sigma^2 + b^2}{(K\sigma + b)^2} \quad \text{for } b > 0. \text{ any!..}$$

choose b; such that RHS minimised...

diff....

$$(K\sigma + b)^2 (2b) - (K\sigma + b)(2)(\sigma^2 + b^2)$$

$$(K\sigma + b)(2)[K\sigma b + \cancel{b^2} - \sigma^2 - \cancel{b^2})$$

$$\boxed{b = \frac{\sigma}{K}}$$

$$\therefore \text{ RHS} = \frac{1}{1+K^2}$$

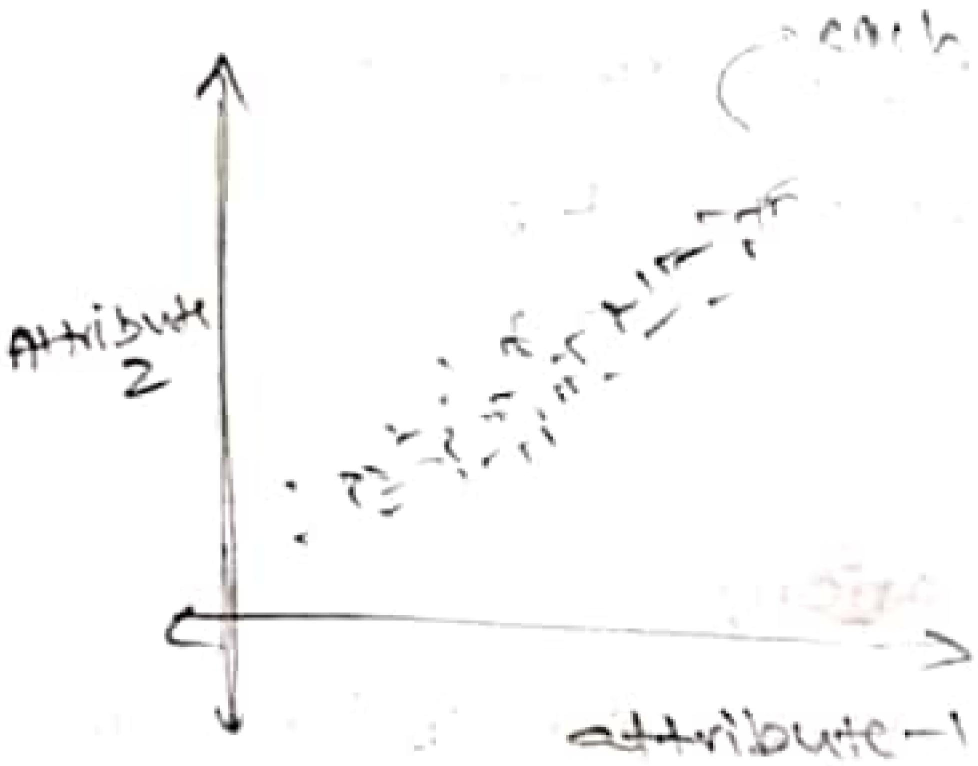
$$\therefore \quad \boxed{\frac{|S_K|}{n} < \frac{1}{1+K^2}}$$

* **corelation b/w different attributes of same sample points:-**

Eg: high fat intake → high heart diease.
high smokes → high cancer rates.

• use scatter plots:-

each (·) is a sample.



} we now, see a corelation na! (able to suggest 'y' based on 'x' & viceversa).

corelation coefficent:-

(x_i, y_i) be sample points.

σ_x, σ_y be standard deviations.

then.

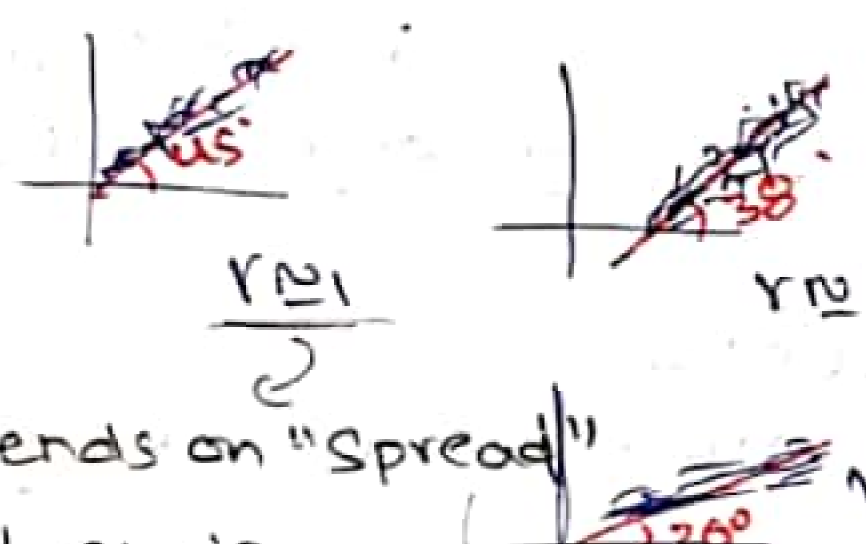
$$\boxed{r(x,y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\left(\sum(x_i-\bar{x})^2\right)\left(\sum(y_i-\bar{y})^2\right)}}} \in [-1,1]$$

cauchy Schwartz inequali

(or)

$$r(x,y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n-1 \cdot \sigma_x \cdot \sigma_y}$$

1) $r>0$: positive corelation.

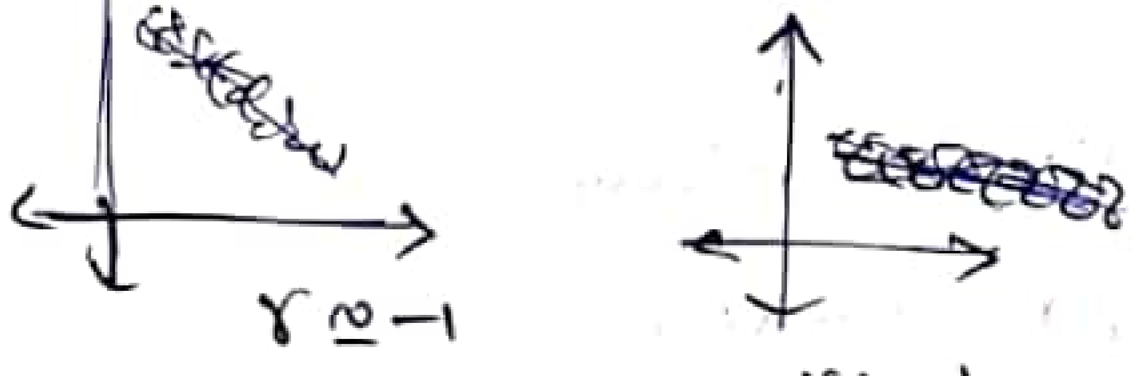


$r \simeq 1$ $r \simeq 1$ $r \simeq 1$

* depends on "spread" not angle, not intercept.

$(\vec{x_i} - \bar{x}\vec{u}) \cdot (\vec{y_i} - \bar{y}\cdot\vec{u})$

2) $r<0$: negative correlation



$r \simeq -1$ $r \simeq -1$

see spread; for magnitude.

$\boxed{\begin{array}{l}\text{Not slope}\\ \text{Not intercept.}\end{array}}$

since $\left.\begin{array}{l}x_i-\bar{x}\\ y_i-\bar{y}\end{array}\right\}$ intercept lost

$\left.\frac{(\quad)}{(\quad)}\right\}$ due to this division, slope lost.

∴ $x, ax+b$
C.F = 1 if $a>0$
C.F = −1 if $a<0$

(PTO).

* take two vectors; in a n-dimension space.

$\vec{x} = x_1\hat{r}_1 + x_2\hat{r}_2 + \cdots x_n\hat{r}_n$

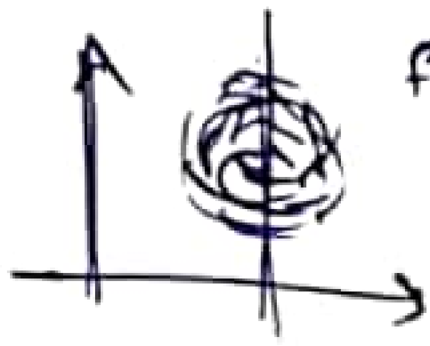
$\vec{x} - (\bar{x})\vec{u} = (x_1-\bar{x})\hat{r}_1 + \cdots (x_n-\bar{x})\hat{r}_n$

$^{\text{ly}}$ $\vec{y} - (\bar{y})\vec{u} = (y_1-\bar{y})\hat{r}_1 + \cdots (y_n-\bar{y})\hat{r}_n$

∴ correlation factor = {angle's cosine} b/w the two "data" vectors.

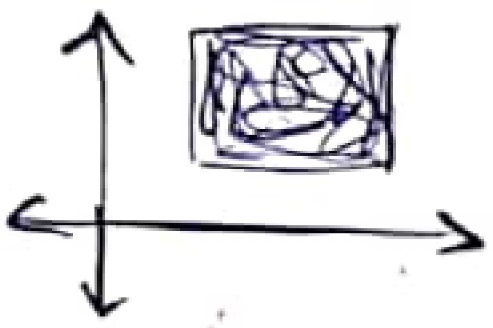
Corelation:- How strongly; I can suggest a unique 'y' for a unique 'x'.

r is defined & 0:-



for an x;
can't at all
suggest a 'y'.

Numerat = 0
deno. $\neq$ 0.

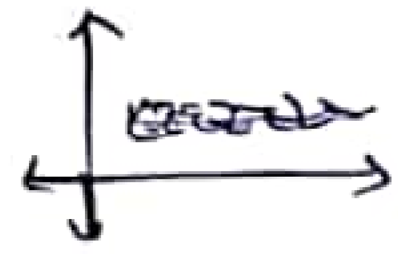


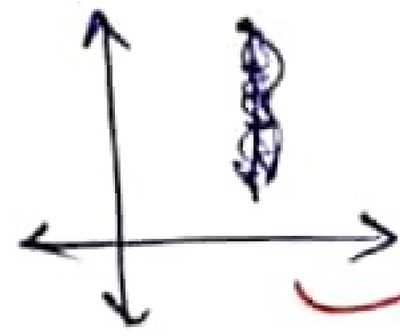
$\Rightarrow$ we can say for sure;

both are 0-correlated.
i.e. no relation b/w both.

r is undefined :-

for x, i can say y,
but for y, i can't say x.





x (no. of ears &
y = salary earned

numerator = 0
denominator = 0

we can't say any co.relation.
not even 0!.

* 

can be certain; in suggesting y!

C.F $\approx$ 1

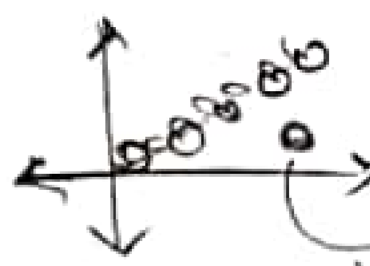
little uncertainity in suggesting y.

C.F > 0; $\neq$ 1
as spread more.

co.relation;
if for some x, i am able to "suggest" y; then I can say co.releation.
How strong I can say, that much co.releation.
So; dosent depend on line's slope.

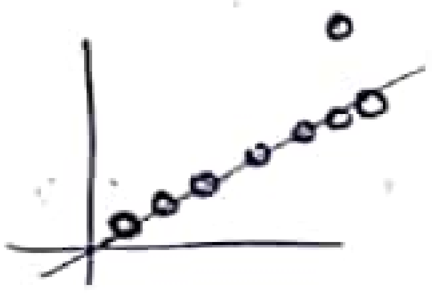
* correlation coefficient - sensitive to outliers.
(error; don't fit on graph).

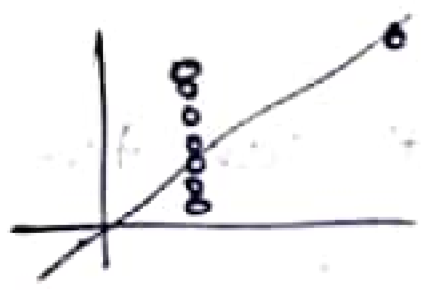
- Ascombe's quartet.



$r = 0.33$
because of 1.

- graphically very different; but same CF.





have same co.creleation coefficient
very misleading!
need to examine graph.

* $$r_{uncentered}(x,y) = \frac{\Sigma x_i y_i}{\sqrt{(\Sigma x_i^2)(\Sigma y_i^2)}}$$ - used somewhere...

* co.oreleation dosent mean causation..

Discrete Probability:- (11, 12th topic).

* B is proper subset of A:

$B \subset A \qquad B \neq A$ & $B \subseteq A$

$\therefore$ B can be NULL.

$\boxed{P(AB) = P(A \cap B)}$

* Boole's inequality:-

$$P(A \cup B) \leq P(A) + P(B)$$

$\therefore$

$$P(A_1 \cup A_2 \cup \cdots A_n) \leq \sum_{i=1}^{n} P(A_i)$$

** Bonferroni's inequality:-

$$P(A \cap B) \geq P(A) + P(B) - 1$$

(loose lower bound; $\because$ no other info, except P(A), P(B))

$\therefore$

$$P(A_1 \cap A_2 \cap \cdots A_n) \geq 1 - n + \sum_{i=1}^{n} P(A_i)$$

* conditional probability:-

$$\boxed{\underbrace{P(A|B)}_{P(A);\ \text{given B happened}} = \frac{P(A \cap B)}{P(B)}} = \frac{P(AB)}{P(B)} = \frac{P(A,B)}{P(B)}$$

notation

* joint probability:-

joint P() of A,B $= P(AB) = P(A \cap B)$ (Nothing!)

$= P(A,B)$

* Independent events:-

$$P(A|B) = P(A)$$
$$\Updownarrow$$
$$P(AB) = P(A) \cdot P(B)$$
$$\Updownarrow$$
$$P(B|A) = P(B)$$

* also A, B^c; A', B'; A', B } are independent

implies they are dependent! mutually exclusive events are not independent! $P(A \cap B) = 0$ but neither $P(A) = 0$ or $P(B) = 0$ $\therefore P(AB) \neq P(A) \cdot P(B)$

for n-events:-

for any 'K'; $\boxed{K \leq n}$ events;

$$\boxed{P(A_1 A_2 \cdots A_K) = \prod_{i=1}^{K} P(A_i)}, K \leq n \ \forall K$$

* only n-way independence doesn't imply events are independent.

Bayes's theorem:-

A,B happens

$$\boxed{P(B/A) = \frac{P(BA)}{P(A)} = \frac{P(A|B)\cdot P(B)}{P(A|B)\cdot P(B) + P(A|B^c)\cdot P(B^c)}}$$

Both

A happens.

AB happens

AB^c happens

B happens; given A happens.

$$\therefore \boxed{P(A) = P(A|B)\cdot P(B) + P(A|B^c)\cdot P(B^c)}$$

infact; in place of B, B^c; you can have ANY number of
- pairwise mutually exclusive
- exhaustive

events.

* Probability of B, given A $=$ $\dfrac{\text{no. of } A\,\&\,B \text{ events}}{\text{no. of } A,B + \text{no. of } A,B^c}$

$\boxed{\text{Simple!}}$

B^c could be

Event C

$\cup$

Event D

$\cup$

$\vdots$

So, more terms in denominator.

Random variable:-

- overview:-
  - discrete & continuous random variables
  - probability mass function (normal wala, for discrete)
    probability density function (pdf)
    cummilative distribution function (cdf)
  - joint & conditional pdf.
  - expectation operator. (a linear operator $E[ax+b] = a \cdot E[X] + E(b)$)
  - variance & covariance
  - Markov's & Chebyshev's inequality.
  - weak law of large numbers
  - Moment generating functions.

* $X, x \to$ R.V.'s value
↳ R.V. name

$P(X=x)$ is a probability mass function.

- continuous random variables:-

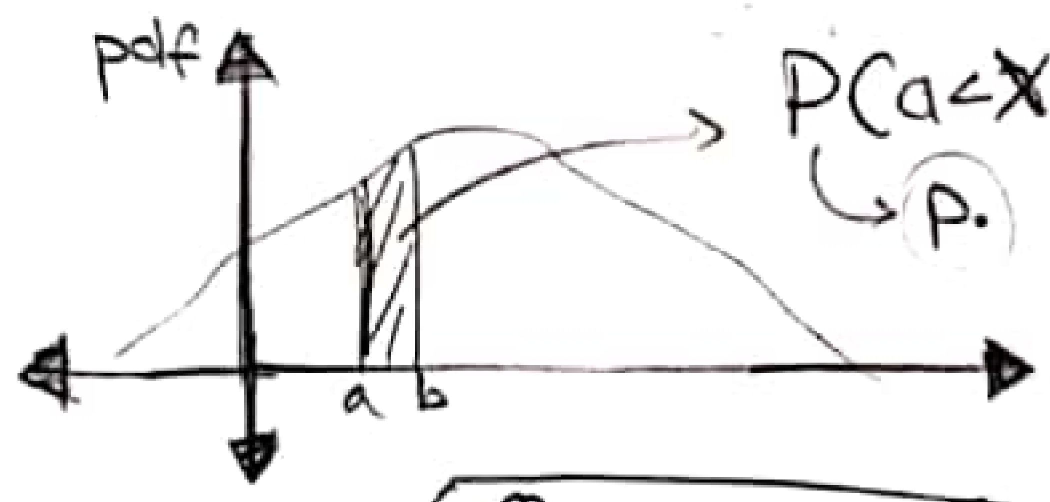
* $P(X=x) = 0$ for any x.
(as, ∞'ly many x are there).

unlike discrete;
$P(X=x)=0$ dosen't imply
$X=x$ dosen't occur.
It may occur!

define $P(X \le x)$ } exists!
(cdf)

$$\boxed{F_X(x) = P(X \le x)}$$

now; $\underbrace{f_X(x)}_{\text{probability density function (pdf)}} = \frac{d}{dx}(F_X(x))$ how much probability in interval 'dx'



$P(a < X \le b) = \int_a^b f_X(x)dx$
↳ P.   P. density
interval probability

$$\therefore \boxed{\int_{-\infty}^{\infty} f_X(x)dx = 1}$$

$$\boxed{F_X(x) = 1 \text{ as } x \to \infty}$$

weighted mean.

expectation value; not same as most probable value!

- (english misused).
- $E(X) \to$ might not exist!
  $P_X(x) = 1/x^2 \cdot k$
- $E(x) \to$ might not be a valid sample point.
  (dice throw; $E(X) = 3.5$)

---

let X be a random var

$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$.

let $g(X)$ be another random varial
inc. funcⁿ y

then
okay; if discrete, but cont.....

LOTUS: $\boxed{E(y) = \int_{-\infty}^{\infty} y \cdot f_X(x) dx}$

law of the unconscious statistisim.

> should hav been:
$E(y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy$

Proof: $y = g(x)$ $F(\,)$ ↳ cdf

$\boxed{F_Y(y) = F_X(g^{-1}(y))}$ common sense.

$\frac{d}{dx}$ on both sides.

$f_Y(y) \cdot \frac{dy}{dx} = f_X(g^{-1}(y)) \cdot \frac{d}{dx}(g^{-1}(y))$

$\therefore \boxed{f_Y(y)dy = f_X(x)dx}$

→ Expected value:

$E[\;]$ → expectation.

- also called mean value of random variable.

$$E(X) = \sum_{i=1}^{n} P(X = x_i) \cdot x_i$$

for discrete.

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot (f_X(x)\,dx)}$$

NOTE: $E[X] \neq \text{mode}(X)$
$E(X) = \text{mean}(X)$
} damn geezers were weak with english.

- $E(X)$ might not even be a valid value of X.
  - might not even exist!.
    (as $\int_{-\infty}^{\infty} x \cdot f_X(x)\,dx \to \infty$).

* Say; i have a new random variable, y

$y = g(X)$.

(equal; if $g(x)$ is linear in x).

then; $E(y)$ need not equal $g(E(X))$

$$\boxed{E(y) = \int_{-\infty} g(x) \cdot f_X(x) \cdot dx.}$$

LOTUS

↓ might seem obvious
but; obvious - only for discrete x;

for continuous X;

$$E(y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) \cdot dy$$

→ we can show both same,
(proof on before page.)

* $E(aX+b) = a \cdot E(X) + b$

b: equal to $E(b)$

Linearity

an "OPERATOR"

* $E(g_1(X) + g_2(X) + \cdots) = E(g_1(X)) + E(g_2(X)) + \ldots$

→ Markov's inequality:-

let $x > 0$ be the possible values of X.

then

$$\boxed{P(X \geq a) \leq \frac{E(X)}{a}}$$

where $a > 0$

very very loose.

proof:

$$E(X) = \int_0^a x f_X(x)\,dx + \int_a^{\infty} x f_X(x)\,dx$$

$$E(X) \geq \int_a^{\infty} x f_X(x)\,dx$$

$$E(X) \geq \int_a^{\infty} a \cdot f(x)\,dx$$

$$\therefore \boxed{P(X \geq a) \leq \frac{E[X]}{a}}$$

weak; usually RHS >> LHS

* mean (expected value) $\longrightarrow$ minimizes the expectation of "squared error" to 0.

minimizes: $\int_{-\infty}^{\infty} (x-c)^2 f_X(x)\,dx$

median $\rightarrow$ minimizes $\int_{-\infty}^{\infty} |x - x_0| \cdot f_X(x)\,dx$

x_0 ↳ a parameter, later called as median.

(do by differentiation)....

$\boxed{F_X(\text{median}) = 0.5}$

cdf.

• variance;

$$\boxed{\sigma^2 = \text{var} = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\,dx.} \rightarrow$$ might not exist; as integral may go ∞.

σ = positive sqrt(var)

* also

$$\text{var} = E((x-\mu)^2)$$
$$= E(x^2 - 2\mu x + \mu^2)$$
$$= E[x^2] - \mu^2$$

* $$\text{var.} = E[x^2] - (E(x))^2$$

cases;

i) mean ✗, variance ✗ } possible

$f_X(x) = \dfrac{K}{x\sqrt{x}}$ { (ii) mean ✓, variance ✗ } possible

supresses mean (x^1) but not variance (x^2)

(iii) mean ✗, variance ✓ } ✗ ∵ mean is part of variance.

* $\text{var}[ax+b] = a^2 \cdot \text{var}[x]$

INEQUALITIES:

* if $x > 0$ then;

$$\boxed{P(X \geq a) \leq \frac{E[X]}{a}} \rightarrow$$ MARKOV's inequality - WEAK

mean.

often: RHS >>> LHS

Proof:

$$E(x) = \int_0^{\infty} x f_X(x)\,dx$$
$$= \int_0^{a} x \cdot f(x)\,dx + \int_a^{\infty} x \cdot f_X(x)\,dx$$
$$> \int_a^{\infty} x f_X(x)\,dx > \int_a^{\infty} a \cdot f_X(x)\,dx$$

$$\therefore \int_a^{\infty} f_X(x)\,dx \leq \frac{E(x)}{a}$$

→ chebyshev's inequality:-

$$\boxed{P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}}$$

proof: take $Y = (X-\mu)^2$ ∴ $Y \geq 0$.

markov's ineq. with $a = k^2$

$\Rightarrow P\{(X-\mu)^2 \geq k^2\} \leq \frac{\sigma^2}{k^2}$] def^n of variance; $\sigma^2 = E((X-\mu)^2)$

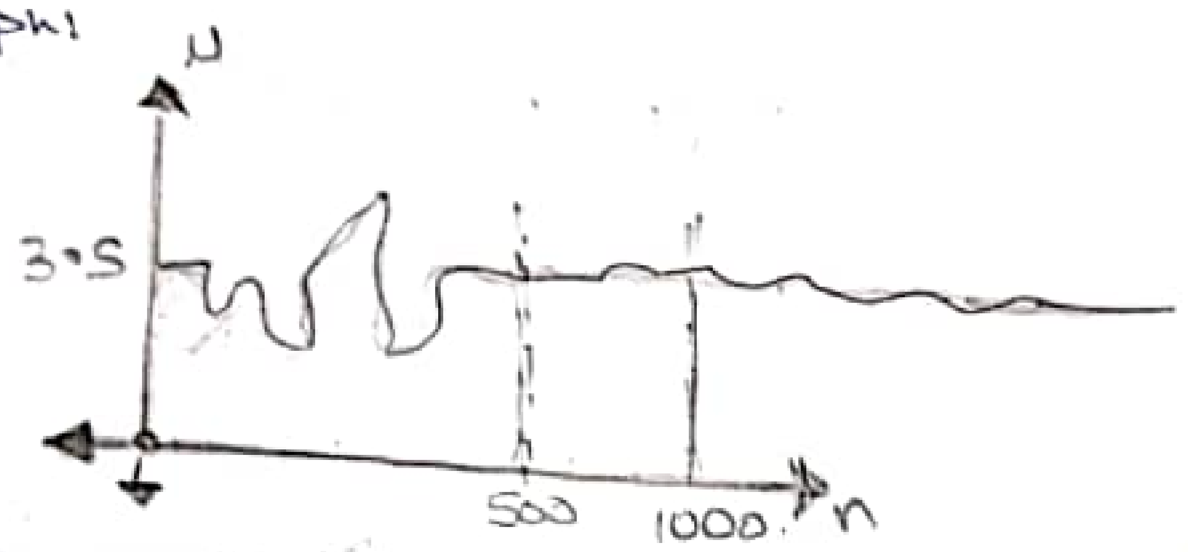
& hence;

$$\boxed{P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}}$$

→ weak law of large numbers:- Good proof, dammit !.

case: I rooll a die n times. what is the average value of die output?

wikipedia graph!



so; why is average 3.5 as n increases?

Ans: You know.... 1 to 6 are equi-probable; so -- = $\frac{1}{6}(1+2+3+4+5+6) = 3.5$ da.

childish argument!

→ Manly arg:-

1) take 'n' dice.

2) X_i = random variable showing output on i^{th} die. → mean = μ = same for all.

so; associate each outcome with a random var.

3) so; now; we need:-

$E\left(\cancel{\frac{X_1+X_2+\cdots X_n}{n}}\right)$. } multiple random variables....

nope!

> we need $P\left(\left\{\frac{X_1+X_2+\cdots X_n}{N}\right\} = \mu\right) = 1$ (not expected value; we need to show; that mean ≡ it is unique value)

Strong law of large nos...

4)

weak law of large numbers:-

let $X_1, X_2, \cdots X_n$ be a seq. of independent & identical distributed random variables; with mean μ (same for all); then for any $\varepsilon > 0$;

$$P\left\{\left|\frac{X_1+X_2+\cdots X_n}{n} - \mu\right| > \varepsilon\right\} \to 0 \text{ as } n \to \infty$$

Proof:- by chebyshevs;

$\left(\because E\left(\frac{X_1+X_2+\cdots X_n}{n}\right) = \mu\right)$

$$P\{| \quad | > \varepsilon\} \leq \frac{var\left(\frac{X_1+X_2+\cdots X_n}{n}\right)}{\varepsilon^2}$$

$$\text{var}\left(\frac{X_1}{n}+\frac{X_2}{n}+\frac{X_3}{n}\cdots\frac{X_n}{n}\right) = \sum_{i=1}^{n} \text{var}\left(\frac{X_i}{n}\right) + \underbrace{\sum_{i=1}^{n}\sum_{j=i+1}^{n} \text{covar}\left(\frac{X_i}{n},\frac{X_j}{n}\right)}_{=0}$$

$$= \frac{1}{n^2}\cdot \underbrace{n\cdot\sigma^2}$$

∵ identical distribution.

$= 0$ for independent events.

↓

can have n^2 term, if not!

$\therefore\ P\{|\ |\geq\epsilon\} \leq \frac{\sigma^2}{n\epsilon^2}$

$\therefore$ as $n\rightarrow\infty$;

$P\{|\ |\geq\epsilon\}\rightarrow 0$ ✓

comments on weak law:

(i) assumption of identical σ is not necessary. but, should be finite.

(ii) Independance is NOT needed;

Just hav to be pair-wise uncorelated.

(i) covar( ) = 0

or

(ii) coeff. of corelation = 0.

5) Strong law of large numbers:-

$$P\left(\lim_{n\to\infty} \frac{X_1+X_2+\cdots X_n}{n} = \mu\right) = 1$$

(weak law; tends to 1)

what shit kadha!

Note:

So; here; finally when we compute μ of 500 dice throws;

$\underbrace{1, 1, 1, 1, 1, 1 \cdots}$

$\mu = 1$

$\underbrace{1, 6, 2, 5, 3, 4, 1, 6, 2, 5, 3, 4, \cdots}$

$\mu = 3.5$

$\underbrace{2, 2, 2, 2, \cdots}$

$\mu = 2.$

probability $= \left(\frac{1}{6}\right)^n$

$\simeq 0$ for large 'n'.

Both are possible values for mean; even after 500 tries

But! $P(\mu = 3.5)$ is tending to 1.

Hence; we write a new random variable; using $X_1, X_2, X_3, \cdots$ (i.e. their mean)

& find $P(\text{new rand\_var} = \mu)$.

might not be high; for small n.

but $\rightarrow 1$; for $n\rightarrow\infty$.

∵ weak law of large numbers.

- (Incorrect) law of averages. (Gambler's fallacy)

coin tossed 20 times $\rightarrow$ 20 heads.

States. ~~so; next time; higher probability for tails.~~

S.S.

next time also, same probability.

$\because$

20 heads + 1 head $\qquad$ 20 heads + 1 tail

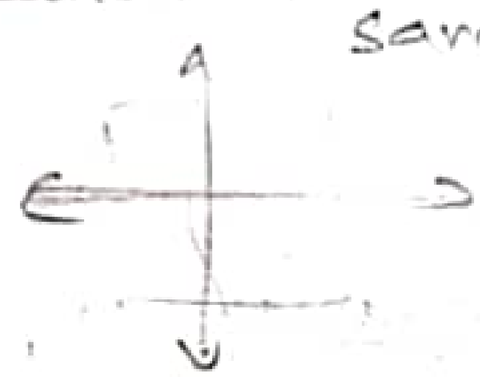
$\left(\frac{1}{2}\right)^{21}$ $\qquad$ $\left(\frac{1}{2}\right)^{21}$

✓✓

Joint Pdfs:- Joint Cdfs:-

* For continuous r.v. X, Y.

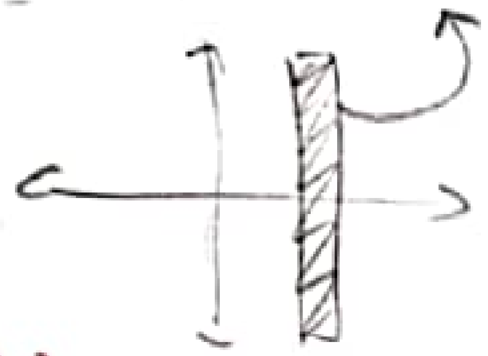
> $\boxed{F_{XY}(x,y) = P\{X \le x, Y \le y\}}$ - 2D plot of distribution of elements in sample space



also

> $F_X(x) = P\{X \le x, -\infty \le Y \le \infty\} = F_{XY}(x, \infty)$

same $F_Y(y)$



"These" definitions are extended; to approach more variables.

→ joint pdf:-

for 1 variable; density over line;

for 2 variable; density over plane.

for probability in a region C:

$$P((x,y) \in C) = \iint_{(x,y)\in C} f_{XY}(x,y)\,dx\,dy. ✓$$

For discrete case:-

$$P(x_i) = \sum_{j=1}^{\infty} P(x_i, y_j)$$

now;

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

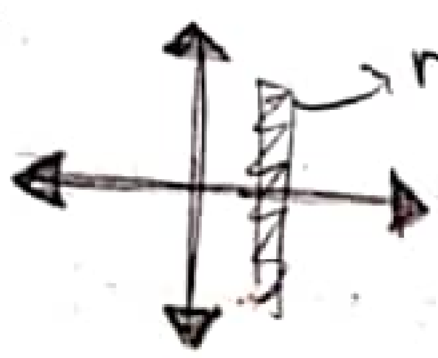
$\therefore$ $\boxed{f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)}$

cool!

→ marginal pdf:-

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$



> Independent Random Variables:-

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

"Joint pdf = product of margin pdfs!!"

& consequently.

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$$

capital means cummilative.

if n such random variables are indpndt then; any K (K≤n) such r.v.s should be "independent"

i.e. joint pdf = product of marginl pdf.

• pairwise independent is necessary; not sufficient.

COVARIANCE:-

product....

> $\boxed{cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]}$ dont write simply $E(X \cdot Y)$!

hence $cov(X,X) = Var(X) = E((X-\mu)^2)$

* $cov(X,Y) = E(XY - \mu_x \cdot Y - \mu_y \cdot X + \mu_x \mu_y)$

$$\boxed{Cov(X,Y) = E(XY) - E(X) \cdot E(Y)}$$

properties:-

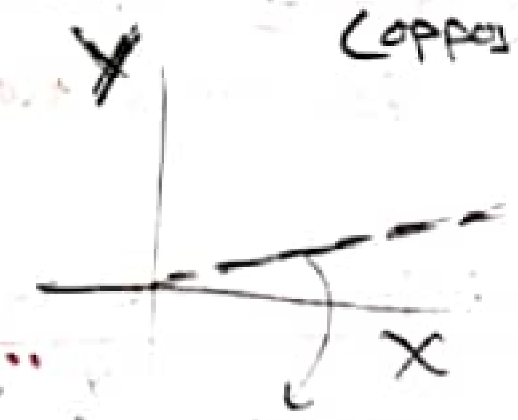
1) $cov(X,Y) = cov(Y,X)$
2) $cov(aX, bY) = a \cdot b \cdot cov(X,Y)$
3) coefficent of corelation:-

$$\boxed{r(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X) \times Var(Y)}}}$$

connect with stuff in discrete statistics..

covariance ≡ co-relat^n

co-variance ✗ independent (opposite)



$r(X,Y) = 1$.

∴ covariance is max.

i.e. fully dependent (X,Y) are....

4) $cov(X+Z, Y) = cov(X,Y) + cov(Z,Y)$

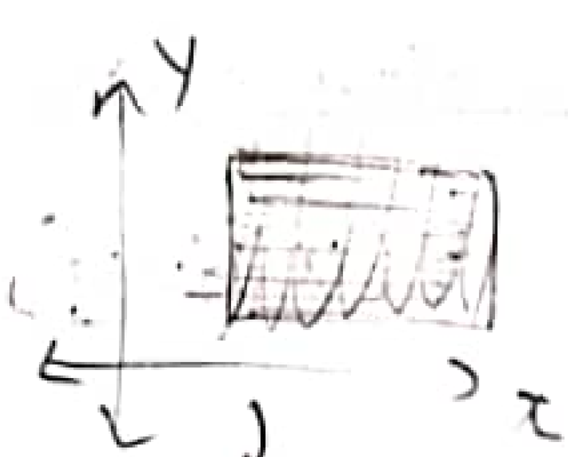
prove from,

$cov(X,Y) = E(XY) - E(X) \cdot E(Y)$

5) $cov\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j cov(X_i, Y_j)$ voray!

6)

$$\boxed{\begin{aligned} var(X_1+X_2+\cdots X_n) &= \sum_{i=1}^{n} Var(X_i) \\ \overset{\shortparallel}{covar}(X_1+X_2+\cdots, X_1+X_2+\cdots) &= \quad + \sum\sum_{i \neq j} covar(X_i, X_j) \end{aligned}}$$



graph; if X,Y are independent corelation = 0.

cov = 0

* for independent events;

$$E(XY) = E(X)\cdot E(Y)$$

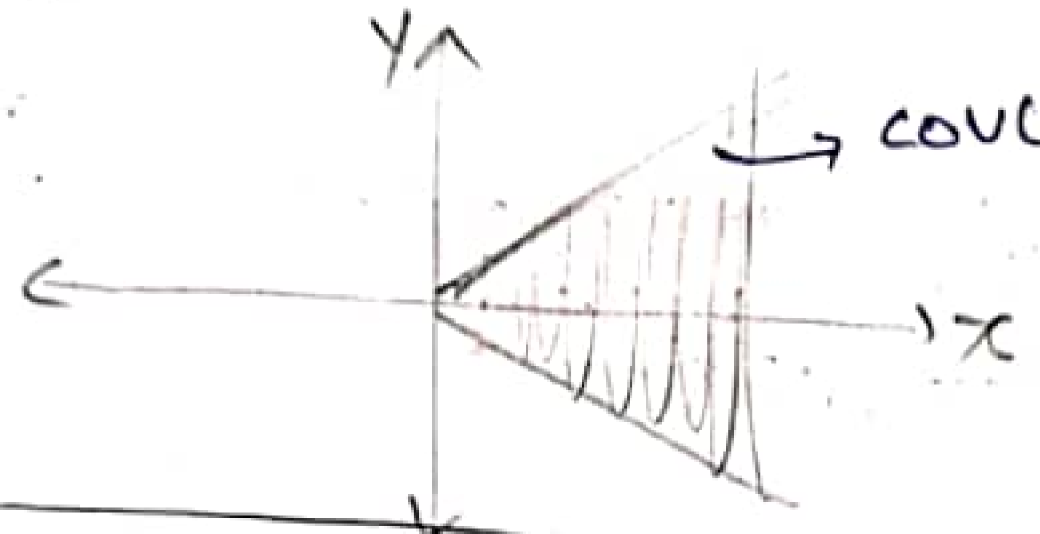
$\therefore$

$\boxed{cov(X,Y) = 0, \text{ for independent events}}$

'co'- variance (varying together) is 0.

but;

$\boxed{\text{if } cov(X,Y)=0 \text{ then independent is wrong.}}$

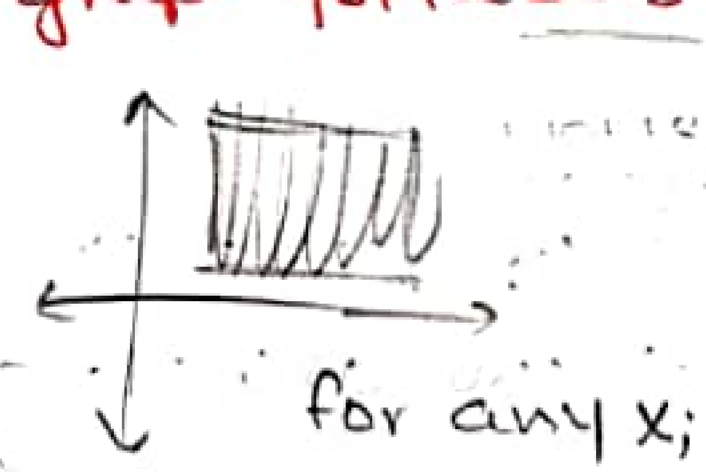


$cov(X,Y)=0$ ($\because$ for every $(X-\mu_X)$; we got $+y_0, -y_0$)

& X, Y are not independent

$\because$ $y \in [-x, x]$

(if X,Y are independent, then graph gotta be sq.)



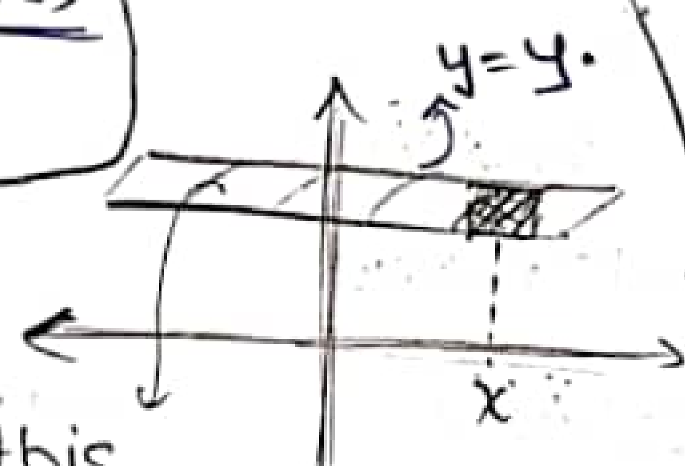
for any x; any y is ok. & vice versa.

→ conditional pdf, cdf:-

joint pdf is $f_{XY}(x,y)$

$$\boxed{\underbrace{f_{X|Y}(x,y)}_{\text{a function in } x.} = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

Y = y is fixed. for conditional.



this line density is $f_{X|Y}(x,y)$ [proof] [vaguely]

$\therefore$ $\boxed{f_{X|Y}(x,y)\cdot dx = \frac{f_{XY}(x,y)\cdot dx\cdot dy}{f_Y(y)dy}}$

points in $(x, x+\delta x)$ & $(y, y+\delta y)$

points in $(-\infty,\infty)$ & $(y, y+\delta y)$

sample space is $(-\infty,\infty)$ $(y, y+\delta y)$

needed is $(x, x+\delta x)$ $(y, y+\delta y)$

• conditional cdf:

$$F_{X|Y}(x,y) = \int_{-\infty}^{x} f_{X|Y}(x,y)dx$$

x, not same r.v.

$$= \frac{\int_{-\infty}^{x} f_{XY}(z,y)dz}{f_Y(y)}$$

✓✓

• conditional mean & variance:-

$$E(X|Y=y) = \int_{-\infty}^{\infty} x\cdot f_{X|Y}(x,y)dx$$

$$var(X|Y=y) = \int_{-\infty}^{\infty} (x - E(X|Y=y))^2\cdot f_{X|Y}(x,y)\,dx$$

take care!

not regular μ_X.

lol.....

Eg: $f_{XY}(x,y) = 2.4\,x(2-x-y)$ $\quad 0<x<1$, $\;0<y<1$

find conditional density of X given Y = y.

Sol)

$$f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_{XY}(x,y)}{\int_0^1 f_{XY}(x,y)\,dx} = \frac{2.4x(2-x-y)}{(2.4)\left(1-\frac{1}{3}-\frac{y}{2}\right)}$$

y = parametric constant.

$$\therefore \boxed{f_{X|Y}(x,y) = \frac{6x(2-x-y)}{4-3y}}$$

MOMENT GENERATING FUNCTIONS:-

* **Moment** of random var. X of order n; is $E(X^n)$.

$\therefore\; m_1 = E(X) = $ mean.

$m_2 = E(X^2)$ → 2nd order moment.

$\boxed{m_i = E(X^i)}$ → i^{th} order moment.

E(X) → mass moment.

moment of inertia → $(E(X^2))$

mass = probability mass.

* Moment generating function: (MGF)

function; to generate various order-moment of X.

$\boxed{\phi_X(t) = E(e^{tX})}$; t is a function parameter.

$$= E\left(1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \cdots\right)$$

$$\phi_X'(t) = E\left(\frac{X}{1!} + 2\cdot t\cdot\frac{X^2}{2!} + \cdots\right)$$

$\phi_X'(0) = E(X) = 1^{st}$ order moment.

$\phi_X''(0) = E(X^2) = 2^{nd}$ order moment.

$\vdots$

$\boxed{\phi_X^{k}(0) = E(X^k)} = k^{th}$ order moment.

↓

moment 'generating' funct^n $\phi_X(t)$.

The MGF is a compact way of encapsulating all ∞ moments of X.

Properties of MGF:-

1) if X,Y are independent;

$$\boxed{\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)}$$

$\iint e^{t(x+y)} \cdot dx \cdot dy \cdot f(x) f(y)$

$= E[A \cdot B] = E(A] \cdot E[B]$

2) $$\boxed{\phi_{aX+b}(t) = e^{tb} \cdot \phi_X(at)}$$

3) let z be x with probability p
& Y with " (1-p)

$$\phi_Z(t) = p \cdot \phi_X(t) + (1-p)\phi_Y(t)$$

→ Uniqueness:- (crazy!) $\phi_X(t)$ generates '∞' moments, which uniquely determine X

* Pdf completely determines a random variable X.

Claim:

> A MGF $\phi(t)$ can uniquely determine a Pmf / Pdf & thus uniquely define a random variable X.

i) case of discrete:

$$MGF = E(e^{tX}) = \Sigma P(x_i) \cdot e^{tx_i}$$

e^{kx} all, are independent vectors kadha.

say; PMF of two random variables X,Y exist

& say $\phi_X(t) = \phi_Y(t)$

compare coeff. of $e^{t \cdot (c)}$ (this is bcoz; every e^{kx} is a base vector..)

$P(X=c) = P(Y=c)$ for any c

like if $a \cdot e^{k_1 x} = b \cdot e^{k_2 x}$ $\forall x$! then $a=b$, $k_1 = k_2$

∴ since $PMF_{X,Y}$ are equal;

$$X \triangleq Y$$

(ii) This uniqueness between MGF & Pdf can be shown for continuous X too!

(formidable proof)

Proof :- ...

X $\quad$ Y

$f_X(x)$ $\quad$ $f_Y(y)$.

$\phi_X(t) = \phi_Y(t)$.

$$\underbrace{\int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx}_{\text{like, do fourier transform for } MGF_X(t)} = \int_{-\infty}^{\infty} e^{ty} \cdot f_Y(y)\,dy \ldots$$

do fourier transform for $\phi_Y(t)$.

$\text{sin}(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$

find $f(x)$ from this fourier transformⁿ...

Families of Random variables.

discrete distribⁿ → continuous distribution.

Discrete distribⁿ:

1) **Bernoulie:** – $\underset{\overset{|||}{1}}{X}$ is head; for a coin toss.

PMF: $P(X=1) = p$; $P(X=0) = 1-p$ — p is parameter, so – family.

that's it.

$* E(X) = p$

$* Var(X) = p(1-p)$ — mode/median $[0,1]$; $\boxed{MGF = (1-p+pe^{t})}$

2) **binomial:**

n–coins; with 'p' for heads, for each coin.

PMF: $P(X=i)$ → i heads exact, among n coins.

$$= {}^{n}C_i \cdot p^i (1-p)^{n-i}$$

$\boxed{n,p}$ parameters.

E(X): let $* X_i$ = bernoulie value for ith coin.

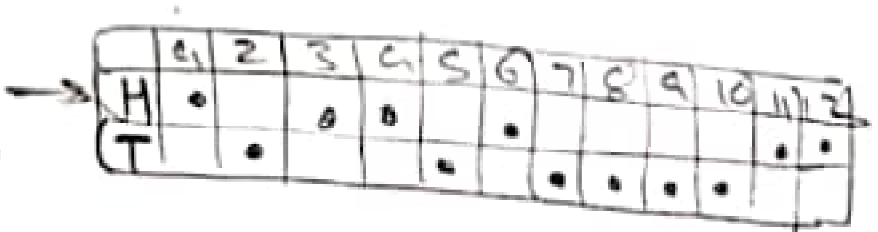
note that:

$$X = \Sigma X_i$$

$$E(X) = E(\Sigma X_i)$$

$$= \sum_{i=1}^{n} p$$

$$E(X) = np$$

X 

$X = 6$ ✓

Var:

$$var(X) = var(\Sigma X_i) = \Sigma Var(X_i) = np(1-p)$$

∵ independence among X_is.

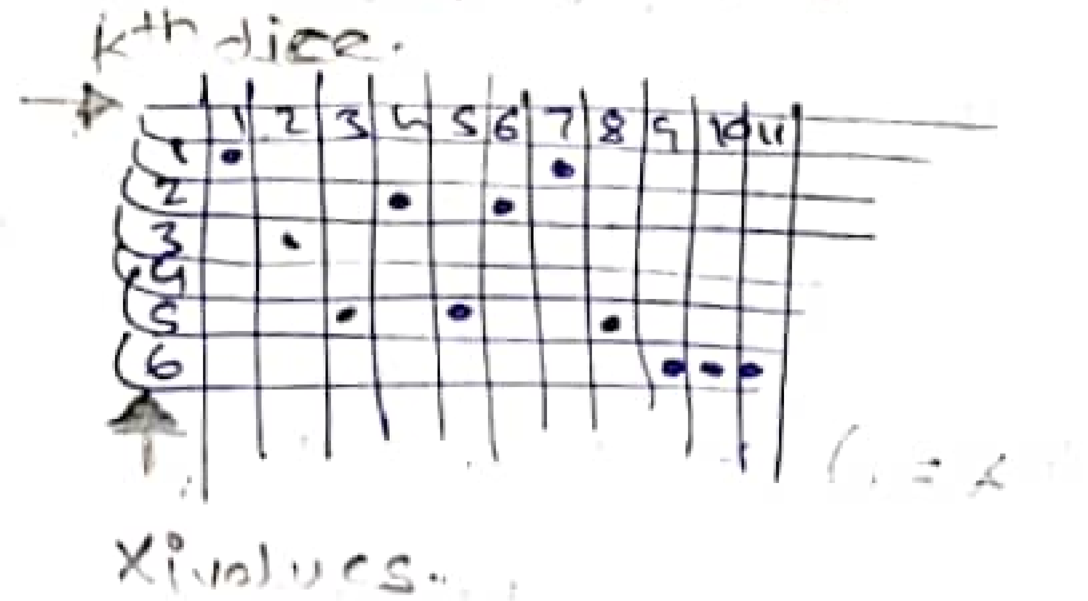
MGF: $= (1-p+p\cdot e^t)^n$

$$\left(\because \phi_X(t) = \phi_{\Sigma X_i}t = \prod_{i=1}^{n} \phi_{X_i}t = (1-p+pe^t)^n\right)$$

! only if independent.

3) Multinomial:-

instead of n-coins; we have n-dices.



- Here random variable X: is vector.

$$P(X=[x_1,x_2,\dots x_n]) = \frac{n!}{x_1!\dots x_n!} \cdot P_1^{x_1} \cdot P_2^{x_2} \dots P_n^{x_n}$$

$(\because \Sigma x_i = n)$; $(\because \Sigma P_i = 1)$.

$E(X)$: $= [E(x_1), E(x_2), \dots E(x_n)]$

$= [E(\Sigma x_{1j}), E(\Sigma x_{2j}), \dots]$

$= [n \cdot P_1, n \cdot P_2, \dots n \cdot P_K]$.

$X_i \rightarrow$ ith possible value of a outcome.

$X_{ij} \rightarrow$ the bernoulle variable; with $P_X(X_i) = P_i$ in, the jth dice throw.

Variance:

$$V(x_i) = Var\left(\sum_{j=1}^{n} x_{ij}\right)$$

$$= n \cdot P_i(1-P_i)$$

* for X; we write covariance matrix

$$= \begin{bmatrix} C(x_1,x_1) & C(x_1,x_2) & \dots \\ C(x_2,x_1) & — & — \\ \vdots & \vdots & \end{bmatrix}$$

note that $C(x_i,x_i) = Var(x_i) = n \cdot P_i(1-P_i)$

$C(x_i,x_j) = -n p_i p_j$] proved in video. okay proof ...

$\therefore$ covar matrix:- symmetric, diagonal positive, remaining negative.

MGF: = take only $P_1, P_2, \dots P_{K-1}$

$$= \left(P_1 \cdot e^{t_1} + P_2 \cdot e^{t_2} + \dots P_{K-1} \cdot e^{t_{K-1}} + 1 - P_1 - P_2 - \dots P_{K-1}\right)^n$$

(we do with $t_1, t_2, \dots t_{K-1}$ to get the vector components).

4) Hypergeometric distribution:-

sampling without replacement.

Say N - goodobjects, M - badobjects ; total pick n - objects.
i.e. $x=1$ $x=0$.

$$P(X=i) = \frac{{}^{N}C_i \times {}^{M}C_{n-i}}{{}^{N+M}C_n}$$; X is hypergeometric r.v.

total; i

$[{}^{a}C_b = 0$, if $b>a$, or $b<0$.]

* $P(X_i) = N/N+M$ $\left(\because P(X_1) = \frac{N}{N+M} \right.$

meaning;
good object came in ith pick.
- a bernoulie R.V.

$P(X_2) = P(X_2=1 \mid X_1=1)\cdot P(X_1=1) + P(X_2=1 \mid X_1=0)\cdot P(X_1=0)$

can tell directly. we've no data.

$= \frac{N}{N+M}$ (on simplificatn)

:

)

* $E(X)$:

$= E(X_1+X_2+\cdots X_n)$

$= n\cdot\frac{N}{N+m}$

$var(X_i)$:

$= \frac{NM}{(N+M)^2}$ (equal to $P_i\cdot 1-P_i$):

* $var(X) = var(X_1+X_2+\cdots X_n)$

$$= \sum_{i=1}^{n} var(X_i) + \sum_{i}\sum_{j\neq i} covar(X_i, X_j)$$

X_i, X_j are not independent

$$= np(1-p)\left[1-\frac{n-1}{N+M-1}\right]$$

covar(X_i,X_j):

$= E(X_i\cdot X_j) - E(X_i)\cdot E(X_j)$

$= P(X_iX_j=1)\cdot 1 + P(X_iX_j=0)\cdot 0 - E(X_i)\cdot E(X_j)$

$= P(X_i=1, X_j=1) - E(X_i)\cdot E(X_j)$

$$= \frac{N-1}{N+M-1}\cdot\frac{N}{N+M} - \left(\frac{N}{N+M}\right)^2$$

$$= \frac{-NM}{(N+M-1)(N+M)^2}$$

Geometric distribution:-

* probability that first heads occurs in the K^{th} trail

$\boxed{f_X(k) = (1-P)^{k-1} \cdot P}$; Here $P(X>K) = (1-P)^K$

memory less

$$\boxed{P(X>s+t) = P(X>s) \cdot P(X>t)}$$

mean $= \frac{1}{P}$.

variance $= \frac{1-P}{P^2}$

II Continuous r.var. distribution :- 1) Gaussian:-

Parameters: mean $= \mu$

std. dev. $= \sigma$

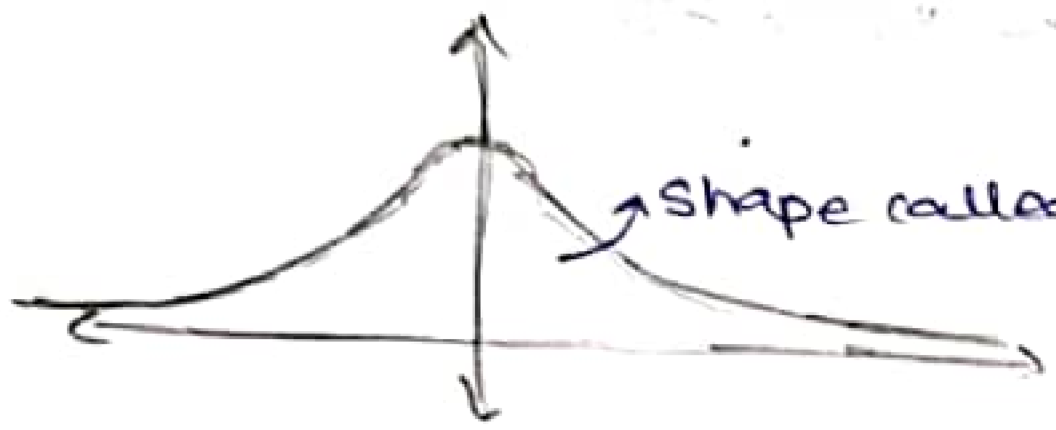
$$\boxed{f_X(x) = \frac{1}{\sigma\cdot\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}} \rightarrow \mathcal{N}(\mu, \sigma^2)$$

Not $\mathcal{N}(\mu, \sigma)$

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\cdot dx = \sqrt{\pi}}$$

proof by taking $\int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy$ (polar)

* $N(0,1) \rightarrow$ standard normal distribution.



shape called as Bell-curve.

mean $= \mu$.

variance $= \sigma^2$

Note:

$X \sim N(\mu, \sigma^2)$;

$aX + b \sim N(a\mu + b, a^2\sigma^2)$

CDF:

$$\phi(x) = F_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-z^2/2} dz \quad \text{for } \mathcal{N}(0,1).$$

(we can't exactly integrate $\int_0^x e^{-x^2} dx$).

* error function $\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot \int_0^x e^{-x^2} dx}$

$\operatorname{erf}(x)$ as $x \to \infty$ $= 1$.

$$* \ \phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \left(\int_{-\infty}^{0} e^{-z^2/2} dz + \int_0^x e^{-z^2/2} dz \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{2}\cdot\sqrt{\pi}}{2} + \sqrt{2} \cdot \int_0^{x/\sqrt{2}} \cdot e^{-z^2} dz \right)$$

$$\phi(x) = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}} \right) \right)$$

$$\phi\left(\frac{x-\mu}{\sigma} \right) = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma} \right) \right)$$

for mean $= \mu$; variance $= \sigma^2$.

* probability for R.V. to be with $\mu - n\sigma$ to $\mu + n\sigma$.

$n=1$	68.2%
$n=2$	95.4%
$n=3$	99.7%
$n=4$	99.9937%

MGF:

$$\phi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

calculate $E(x)$ ✓

$E(x^2)$ ✓

Proof:

$$\int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}} \cdot dx.$$

$$e^{tx} \cdot e^{-x^2/2} \frac{1}{\sqrt{2\pi}}$$

$$e^{t^2/2} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx$$

$$e^{t^2/2} \sqrt{2\pi}.$$

$$\frac{x-\mu}{\sigma} = K.$$

$$x = \sigma K + \mu.$$

$$e^{\mu}$$

$$\sigma \cdot dK.$$

$$e^{t(\sigma K + \mu)}$$

$$e^{t\mu} \cdot e^{t\sigma K} \cdot e^{-\frac{K^2}{2}} dK.$$

$$e^{\frac{(\sigma t)^2}{2}}$$

* Central limit theorem:-

carry; n rounds of an experiment's random variables be

$X_1, X_2, \ldots$ are iir (identical, independent random var.)

— — — — — — — — —

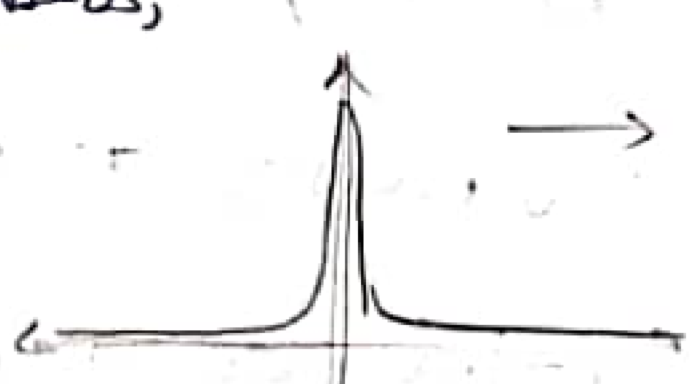
$x_1 \; x_2 \; x_3 \; ---- \; x_n.$

mean $= \mu$;

(i) if $y = \frac{\Sigma x_i}{n} - \mu$:-

By weak law of large numbers;

as $n \to \infty$; $y \to 0$.



$\longrightarrow$ Gauss: $N(0, \to 0)$

(ii) say $y = \sqrt{n}\left(\frac{\Sigma x_i}{n} - \mu\right)$:-

y itself will be a gaussian Random variable $N(0, \sigma^2)$ irrespective of the original distribution of x_i.

i.e. > for a single round; take n-experiments. (iid)

$X_1, X_2, \dots X_n$ ↳ (μ, σ^2) each X_i

& take their mean $(= y)$

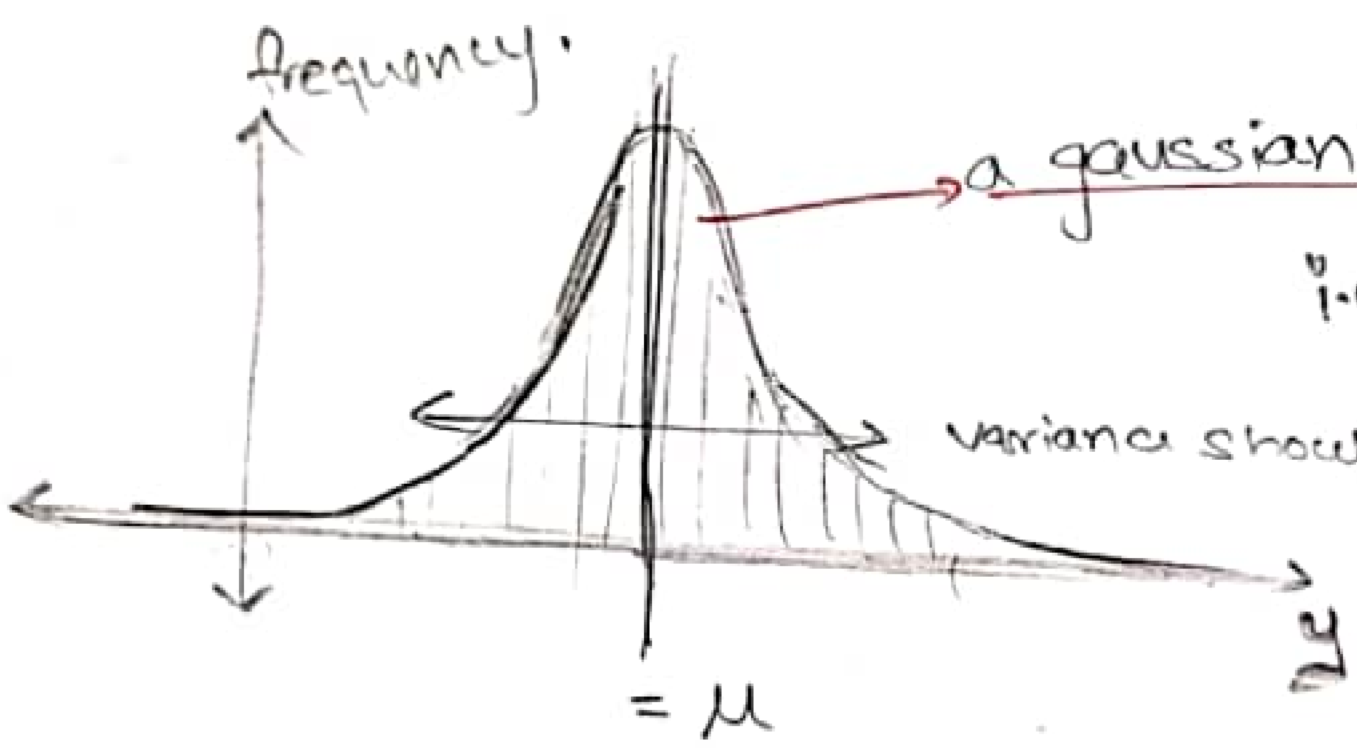
> do 't' such rounds

$$y_1 = \frac{(X_1)_1 + (X_2)_1 + \dots (X_n)_1}{n}$$

$$y_2 =$$

$$y_t =$$

> now histogram of all such y:



→ a gaussian dist-!

i.e. Y is a gaussian RV.

variance should be $= \frac{\sigma^2}{n}$

* statement:-

Consider $X_1, X_2, \dots X_n$ to be a sequence of independent & identically distributed r.v.s with each, having

mean $= \mu$ $(\neq \infty)$

Variance $= \sigma^2$ $(\neq \infty)$ (no where talked about actual pdf (x_i)) so; CLT is powerful.

then the distribution of

$$Y_N = \sqrt{n}\left(\frac{\Sigma X_i}{n} - \mu\right)$$

converges to $N(0, \sigma^2)$ as $n \to \infty$.

Lindeberg–Levy central limit theorem.

* one version of CLT; requires only independence of $X_1, X_2 \dots$

then $Y_n = \left(\frac{\Sigma(X_i - \mu_i)}{\Sigma \sigma_i^2}\right)$ is gaussian variant.

with some condn ( ) → 0.

out of portion?

(i.e. can have different distributions too!!)

* this, more general version of CLT is lindeberg's CLT.
  - provides major motivation of widespread use of gaussian distn.
  - errors in experiments are thus modelled gaussian.

* NO disparity b/w CLT & law of large numbers...

$\boxed{\text{gaussian } \mathcal{N}(\mu, \sigma)}$

* proof:-

consider $z = \left(\frac{\sum x_i - \mu n}{\sigma\sqrt{n}}\right)$

z is $\mathcal{N}(0,1)$ as $n \to \infty$.

• we do this by finding MGF(Z) as $n \to \infty$.

$$\phi_z(t) = \left(\phi_{x-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$

— use facts
i) $\phi_{x+y}(t) = \phi_x(t) \cdot \phi_y(t)$ for independent.
(ii) $\phi_{ax+b}(t) = e^{bt} \phi_x(at)$

we got to prove:

$$\lim_{n\to\infty} n \cdot \log\left(\phi_{\tilde{x}}\left(\frac{t}{\sigma\sqrt{n}}\right)\right) = \frac{t^2}{2}$$

Bcoz MGF$\left(\mathcal{N}(0,1)\right) = e^{t^2/2}$

$n = \frac{1}{x^2}$ (substn) for lim. calculation.
& L-hopital rule 2-times. ✓.

$$\therefore \boxed{\phi_z(t) = e^{t^2/2}}$$

Now; By uniqueness of MGF;

$$\because \phi_{\mathcal{N}(0,1)}(t) = e^{t^2/2}$$

$$\therefore z \equiv \mathcal{N}(0,1)$$

Hence proved CLT — simpler version.

• Gaussian Tail Bounds:-

$CDF_X(X \geq x)$ ke liye, upper bound.
tail probability. not exact $P_X(X \geq x)$

$$P(X \geq x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} dt \leq \int_x^{\infty} \frac{t}{x\sqrt{2\pi}} e^{-t^2/2} dt$$

$\mathcal{N}(0,1)$ for $x>0$

we get:

$$P(X \geq x) \leq \frac{e^{-x^2/2}}{x\sqrt{2\pi}}.$$

* Binomial dist. to Gaussian dist:-

* for large n; binomial dist. approaches to Gaussian dist. with appropriate mean & var.

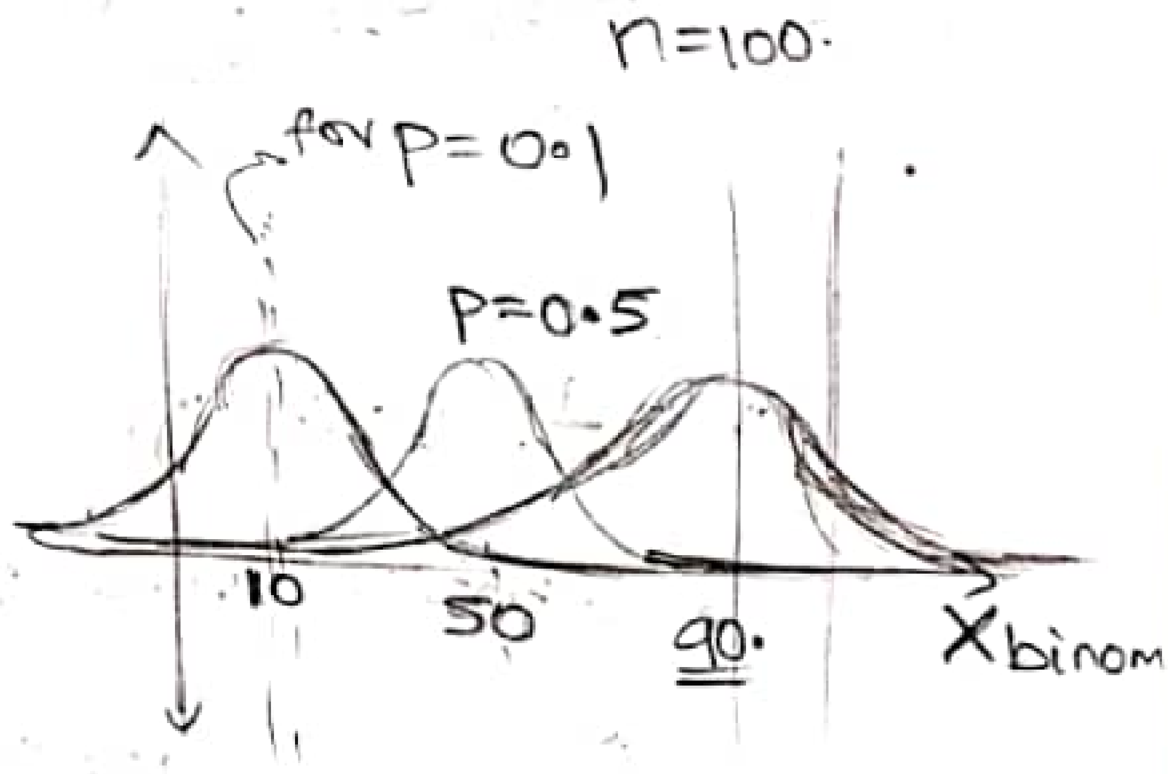
say n.

then $X_{binomial} = \underbrace{X_1}_{\text{Bernoullie...}} + X_2 + \cdots X_n$

$E(X_{bino}) = n \cdot p.$

$Var(X_{bino}) = n(p \cdot (1-p))$

$$Y = \frac{X_{bino} - n \cdot p}{\sqrt{n} \cdot \sqrt{(p)(1-p)}}$$



Y is $\mathcal{N}(0,1)$ for $n \to \infty$

$X_{bino} \sim Binomial(n, p)$

$\therefore X_{bino}$ is $\mathcal{N}(np, n^2 p(1-p))$

for sufficently Large n. (50) ✓

Haha.
whether see from bino point of view
(or) Gauss point of view

mean = np
Var. = np(1-p) $\Big\}$ won't change na

I) sample mean:-

consider 'n' i.i.r $X_1 X_2 \dots X_r$ with μ, σ^2

then $\bar{X} = \frac{\Sigma X_i}{n}$ is a random variable. Called the sample mean.

experimental rigorous

1) By Law of Large numbers:-

as $n \to \infty$ $\quad E(X) = \mu$

$P(X == \mu) \to 1.$ $\quad Var(X) = \frac{1}{n^2}(n \cdot \sigma^2) = \frac{\sigma^2}{n}.$

2) By CLT:-

for sufficiently big n;

X is a gaussian dist. (approximately)

$X \sim N(\mu, \frac{\sigma^2}{n})$

Note:-

HW-2 problem.

if $X_1, X_2, \dots X_n$ were independent normal random (Gaussian) variable

* didn't say identical.

then $X = \frac{\Sigma X_i}{n}$ is also a normal (Gaussian) random variable.

(No need of CLT or $n \to \infty$).

Proof: By MGF:

$$\phi_X(t) = \phi_{\frac{X_1}{n} + \frac{X_2}{n} + \dots \frac{X_n}{n}}(t)$$

$$= \phi_{X_1}\left(\tfrac{t}{n}\right) \cdot \phi_{X_2}\left(\tfrac{t}{n}\right) \dots$$

$$= e^{\frac{t\mu_1}{n} + \frac{t^2}{2}\frac{\sigma_1^2}{n^2}} \cdot e^{t\mu_2 + \frac{t^2}{2}\sigma_2^2} \dots$$

$$= e^{t\left(\frac{\Sigma \mu_i}{n}\right) + \frac{t^2}{2}\left(\frac{\Sigma \sigma_i^2}{n^2}\right)}$$

$$= e^{t(\mu_{net}) + t^2/2\left(\frac{\Sigma \sigma_i^2}{n^2}\right)}$$

By uniqueness theorem (1 MGF $\longleftrightarrow$ 1 pdf)

$X \sim N(\mu_{net}, \sigma_{net}^2)$

$X \sim N\left(\frac{\Sigma \mu_i}{n}, \frac{\Sigma \sigma_i^2}{n^2}\right)$

II) Sample variance:- (S^2)   experimental variance.

$X_1, X_2, \ldots X_n$ are iirs.

then

$$S^2 = \frac{\sum_1^n (x_i - \bar{x})^2}{n-1} = \frac{\sum x_i^2 - n\cdot\bar{x}^2}{n-1}$$

now;

$E(S^2) = ?$ from a x_i; we are having n instances x_1, x_2, x_n

$$= \frac{1}{n-1}\cdot\left(\sum E(x_i^2) - n\cdot E(\bar{x}^2)\right)$$

$$= \frac{1}{n-1}\left(\sum\left[\text{var}(x_i) + E(x_i)^2\right] - n\left[\text{var}(\bar{x}) + E(\bar{x})^2\right]\right)$$

their variance should be σ^2 na...
variance of distribution

$$= \frac{1}{n-1}\left(n\cdot\sigma^2 + n\cdot\mu^2 - n\frac{\sigma^2}{n} - n\mu^2\right)$$

$$= \frac{\sigma^2\cdot(n-1)}{n-1}$$

$$= \sigma^2.$$

$\therefore$ "expectation" of their variance should obviously be σ^2. lets see.

Note:

Here, we used $S^2 = \frac{\sum_1^n (x_i - \bar{x})^2}{n-1}$; rather than $S^2 = \frac{\sum (x_i - \bar{x})^2}{n}$

something related with unbiased estimator.

would give $E(S^2) = \frac{\sigma^2(n-1)}{n}$

$\therefore$

expected value of sample variance of n trials outcomes $\neq$ true variance of trials.

this is undesirable.

so; we multiply S^2 with $n/n-1$

(Bessel's correction)

* what about 'distribution' of sample variance. (i.e. S^2)?

· learn a new distribution.

Huff...

---

2) Chi-square distribution:-

- $z_1, z_2, \dots z_n$ are independent, standard normal random var.s, $N(0,1)$

then

$$X = z_1^2 + z_2^2 + \dots z_{n-1}^2 + z_n^2$$

(as $n \to \infty$; X should tend to gaussian. - CLT)

X is a chi-square random variable. with 'n' degrees of freedom.

woah.

$\boxed{X \sim \chi_n^2}$ chi...

$$\boxed{f_X(x) = \frac{x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}}{2^{n/2} \cdot \Gamma(n/2)}}$$

gamma function.

$$\Gamma(t) = \int_0^\infty x^{t-1} \cdot e^{-x} dx$$

$$\Gamma(n/2) = \frac{n}{2} \cdot \Gamma\left(\frac{n}{2}-1\right)$$

$$\Gamma(1/2) = \sqrt{\pi}.$$

if $n = 2K$;

$$\Gamma(n/2) = K!$$

* deriving for $n=1$.

$$X = Z_1^2 \; ; \; Z_1 \sim N(0,1)$$

$$F_X(x) = P(Z_1^2 \le x)$$

$$= P(Z_1 < \sqrt{x}) - P(Z_1 < -\sqrt{x})$$

$$= F_{Z_1}(\sqrt{x}) - F_{Z_1}(-\sqrt{x})$$

$$\therefore f_X(x) = \frac{1}{2\sqrt{x}} \left(f_{Z_1}(\sqrt{x}) + f_{Z_1}(-\sqrt{x}) \right)$$

$$= \frac{1}{2\sqrt{x}} \cdot \frac{e^{-x/2}}{\sqrt{2\pi}} \times 2$$

$$\therefore \boxed{f_X(x) = \frac{x^{-1/2} \cdot e^{-x/2}}{2^{1/2} \cdot \Gamma(1/2)}}$$

MGF; for X_n^2 is:-

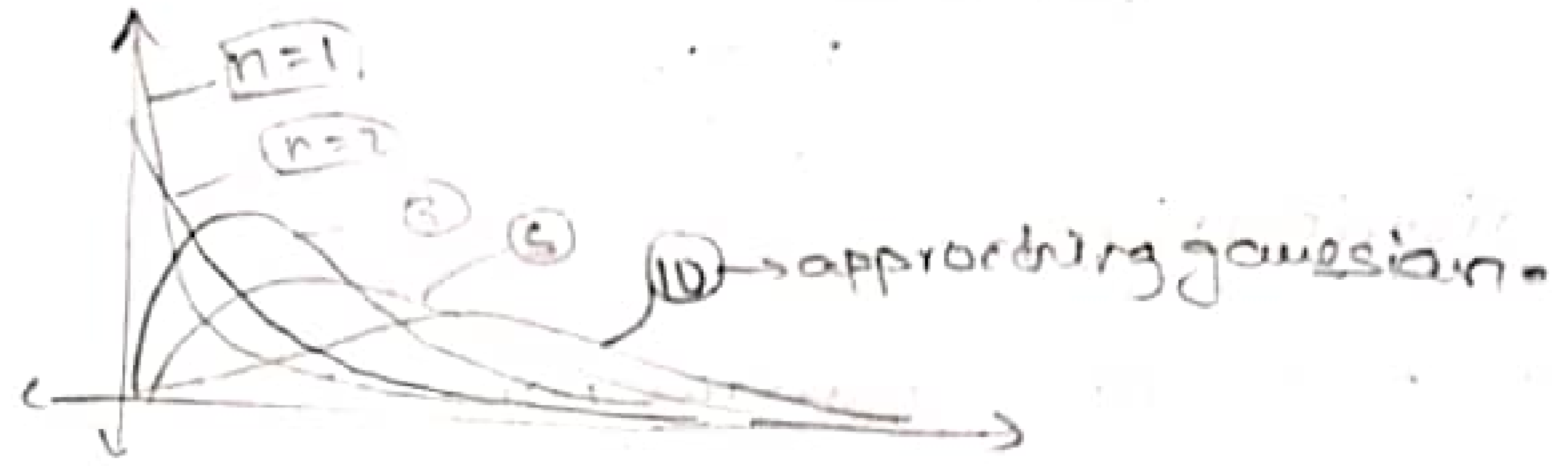
$$\phi_{X_n^2}(t) = (1-2t)^{-n/2}$$

(do for $\phi_{X_1^2}(t)$ & multiply 'n' times)

↓

Now; we can find, or "verify" $f_{X_n^2}(X)$ ✓

<plot of X for different n; vs x>



- Additive property:-

$X = U_1 + U_2$

chi-sq. R.V. with n_1 deg. of freedom; U_2 with n_2

then X is chi-sq. R.V. with (n_1+n_2) df.

---

* now; $S^2 = \dfrac{\Sigma x_i^2 - n(\bar{X})^2}{n-1} = \dfrac{\Sigma (x_i - \bar{x})^2}{n-1} = \dfrac{\Sigma (x_i - \mu)^2 - n(\bar{x}-\mu)^2}{n-1}$

(variance of $X = x+b$).

$$\therefore \underbrace{\Sigma \left(\frac{x_i-\mu}{\sigma}\right)^2}_{\text{complete is a sum of 'n' standard Normal R.V. (}x_i\text{ is normal R.V.)}} = \frac{\Sigma (x_i-\bar{x})^2}{\sigma^2} + \underbrace{\left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}\right)^2}_{\text{via CLT; this is std. gaussian R.V.}}$$

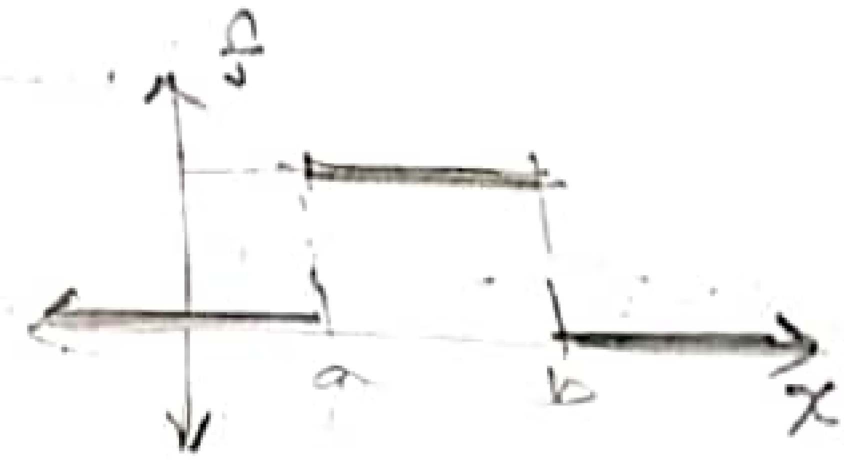
χ^2_n $\qquad$ χ^2_1

Both are independent? $\boxed{\text{Yes!}}$

$\therefore \frac{\Sigma (x_i-\bar{x})^2}{\sigma^2}$ is a χ^2_{n-1} R.V.

3) Uniform distribution:-

$$f_X(x) = \frac{1}{b-a}, \quad a<x<b$$
$$0, \quad \text{otherwise.}$$



$\star\ \mu(X) = \frac{b+a}{2}$

$\star\ Var(X) = \frac{(b-a)^2}{12}.$

$$MGF = \int_a^b e^{tx} \cdot \frac{1}{b-a} = \frac{e^{bt} - e^{at}}{t(b-a)} \quad \text{for } t \neq 0.$$
$$= 1. \quad \text{for } t=0.$$

* Application:-

i) if we somehow bring uniform dist. R.V. X; then we can prepare samples for other distributions.
Like Gauss, poison, ---

ii) say;

$P(0) = 0.3 \quad P(1) = 0.3 \quad P(2) = 0.4.$

now...

$X \sim$ uniform $[0,1]$

the range of probability value.

if $(0 \le X < 0.3)$: sample value $= 0$.

if $(0.3 \le X \le 0.6)$: value $= 1$.

if $(0.6 \le x \le 1)$: value $= 2$.

(iii) describe the working of randperm:-

- i.e. select a subset of Length k; from a set of Length n.

Like;

bcoz; $\boxed{P(I_j \mid I_1, I_2, \cdots I_{j-1} \text{ are known}) = \frac{k - \sum_{1}^{j-1} I_i}{n-(j-1)}}$

now pick X from uniform $[0,1]$

if $\quad X < P(I_j \mid \cdots); I_j = 1$

else $\quad I_j = 0$

4) poission distribution:-

this is discrete R.V. case.

* A binomial dist. where $E(X)$ is fixed; is discussed as poission dist.
& $n \to \infty$

$$\text{Pmf}(X=i) = \frac{n!}{(n-i)!\cdot i!}\cdot (P)^{i}\cdot(1-P)^{n-i}$$

$\because n\to\infty$
& $E(X) = np = \lambda$.
$\therefore p = \frac{\lambda}{n}$.

$$P(X=i) = \frac{\lambda^i}{i!}\cdot\frac{n!}{(n-i)!\cdot n^i}\cdot\underbrace{\boxed{(\sim 1)^{n-i}}}_{(1-\frac{\lambda}{n})^{n-i}}$$ wait!

$$\simeq \frac{\lambda^i}{i!}\cdot\boxed{\left(1-\frac{\lambda}{n}\right)^n}$$

not 1.
this is $e^{-\lambda}$.

$$\boxed{P(X=i) = \frac{e^{-\lambda}\cdot\lambda^i}{i!}}$$

poission Pmf
not Pdf

- sample size is large $(n\to\infty)$ but $E(X)$ is finite. $=\lambda$.

$\therefore \sum_{i=0}^{\infty} P(X=i) = 1$.

* $E(X) = \lambda$
* $Var(X) = \lambda$

think of binomial; with $np=\lambda$ & $p\to 0$

* Mgf: $\sum e^{-\lambda}\frac{(e^t\lambda)^i}{i!} = e^{-\lambda}(e^{e^t\lambda})$

$$\boxed{\phi_X(t) = e^{\lambda(e^t-1)}}$$

weirdest.

- for larger λ; Poission$(\lambda) \sim N(\lambda,\lambda)$

* Say $z = X + Y$.

where X is poission(λ_1)

Y is poission(λ_2)

& both are independent.

then $Z = \text{poission}(\lambda_1 + \lambda_2)$ proof by considering:-

$\phi_z(t) = \phi_x(t) \cdot \phi_y(t)$.

* $\dfrac{PMF(i+1)}{PMF(i)} = \dfrac{\lambda}{i+1}$

* if $X \sim$ poission (λ)

& $P(Y-| X=\eta) = \text{Binomial}(\eta, p)$

η of Bino. dit.

pot Bino. dit

then

$Y \sim$ poission(λp)

thinning of poission rand. var. by a binomial.

Pmf $(y = x_0)$
$= \Sigma P(X=\ell) \cdot X_{B(\ell,p)}(x_0)$

$P_i \quad e^{t \cdot P_i} \cdot e^{t} \cdot e$

$P_i \cdot x_i \quad 1-p$

practical use:-

Points in a scene being imaged, send out photons at rate of λ (poisson).

of these; only a small frac. p, managed to enter the camera (binomial)

$\therefore$ effective rate captured by camera is λp (poission)

$= \sum_{\ell=} \dfrac{e^{-\lambda}\lambda^{\ell}}{\ell!} \times \dfrac{\ell!}{(\ell-x)!\,x!} \cdot p^{x}(1-p)^{\ell-x}$

poisson eg:-

fixed no. of occ. per day ---
accidents...

but police watches every 10 min.

5) Exponential distribution:-

* say; a process has 'λ' chances to succeed in unit time.

  $\therefore$ '$\lambda \cdot u$' chances; in 'u' time.

  poisson process

  let T denote the time; when first success occurs

  – waiting time.

* we have

  T = a random variable.

  $F_T(T \leq u) = 1 - P(T \geq u)$

  meaning;
  in time 'u';
  '0' successes.
  $= Pmf_{poiss(\lambda u)}(=0)$
  $= e^{-\lambda u}$

  $\boxed{P(T > u) = e^{-\lambda u}}$

  imp. distinct feature of exp. rand. var.

> $\therefore \boxed{F_T(u) = 1 - e^{-\lambda u}}$ what the shit is going on here

> $\therefore \boxed{f_T(u) = \lambda \cdot e^{-\lambda u}}$ $u \geq 0$.

exponential random var.

* $E(T) = \frac{1}{\lambda}$ $\left(= \int_0^\infty t \cdot \lambda \cdot e^{-\lambda t} dt = \lambda \cdot \left[\cancel{\left[\frac{t \cdot e^{-\lambda t}}{-\lambda} \right]_0^\infty} + \left[\frac{e^{-\lambda t}}{-\lambda^2} \right]_0^\infty \right] \right.$

  0 $\quad = \frac{1}{\lambda^2}$

  $1/\lambda$

* $Var(T) = \frac{1}{\lambda^2}$

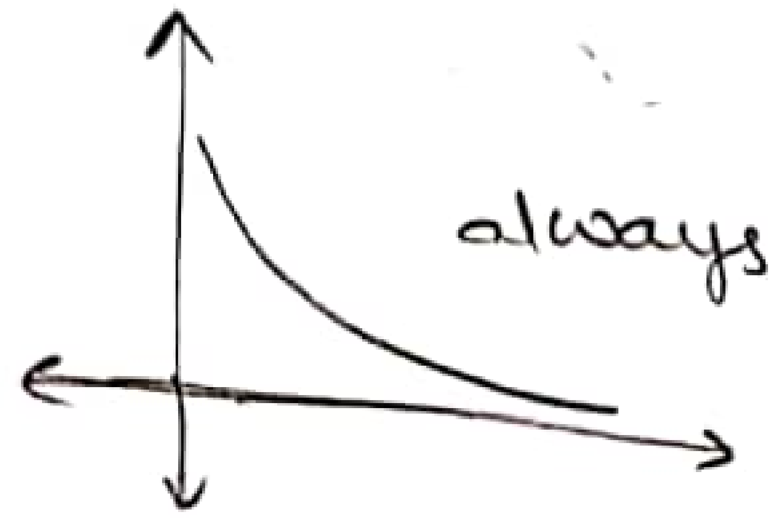
* $MGF(T)(t) = \frac{\lambda}{\lambda - t}$ } whatever.

  waw,

* mode is $u = 0$

* median: $\int_0^{x_0} f_T(u) \cdot du = 1/2$

  $\boxed{x_0 = \frac{\ln 2}{\lambda}}$



always exponential decay.

* This exp. R.V. is said to be memoryless:-

$\forall s,u \geq 0 \quad P(T > s+u \mid T > u) = \underline{P(T > s)}$

This is not independence-101---

proof:

$P(T > s+u \mid T > u)$

$= \frac{P(T > s+u, T > u)}{P(T > u)}$

$= \frac{e^{-(s+u)\lambda}}{e^{-(u)\lambda}}$

$= e^{-s\lambda}$

$= P(T > s)$

$\because$ (exponential tail).

also

We haven't got 2 variables here.

RHS $\neq P(T > s+u)$

* if $x_1, x_2, \ldots x_n$ are exp. rand. var;

$X = \min(x_1, x_2, \ldots x_n)$ is also a exp. rand. var

$\because \quad P(X > x_0) = P(X_1 > x_0 \cap X_2 > x_0 \cap \ldots)$

$= e^{-\lambda_1 x_0} \cdot e^{-\lambda_2 x_0} \ldots e^{-\lambda_n x_0}$

$$\boxed{P(X > x_0) = e^{-(\lambda_{net}) x_0}}$$

so,

$F_X(x_0) = 1 - e^{-\lambda_{net} x_0}$

$\therefore f_X(x_0) = \lambda \cdot e^{-\lambda_{net} x_0}$

Self-

* to find $f_X(x)$ for some distribution; (methods)

Like $\min(X_1, X_2 \ldots)$
or just $X_1 + X_2 + \cdots$

(i) maybe multiply individual $f_{X_1}(x_1), f_{X_2}(x_2) \ldots$

(for independent variables & joint probability).

(ii) Find CDF first. & get pdf from that.

≡ counting manually in case of continuous distributions

$y = \min(X_1, X_2, X_3, \ldots)$ or $y = \max(X_1, X_2 \ldots)$

then

$$P(Y \le y) = 1 - P(Y > y)$$

$$= 1 - P(X_1 > y) \cdot P(X_2 > y) \cdots$$

MLE of uniform distribut^n.

$\therefore$ now; $\text{pdf} = \frac{\partial}{\partial y} P(Y \le y)$

discrete case.
|||

(iii) Start counting manually. (we do this for basic distributions) AND discrete r.v..

like $f(X = x) = {}^nC_x \cdot p^x \cdot (1-p)^{n-x}$.

(binomial).

(iv) Look at MGF:

Thing

(if MGF is of any one of known forms; the by uniqueness theorem; we can prove $f_X(x)$.

(used in CLT proof,

also used in proving

$X = X_1 + X_2 + \cdots X_n$

& each X_i is gaussian

then X is gaussian).

Self :-

* to find $E(X)$ & $var(X)$ for some random var. X :-

(i) do the summation directly :-

$$E(X) = \int x \cdot f_X(x) dx$$
$$(or) \sum x \cdot f_X(x)$$
$$var(x) = E((X-\mu)^2)$$

} used very less; in case of the standard distributions.

(ii) Try to write X as $X_1 + X_2 + X_3 + \cdots$:-

now;

$E(X) = E(X_1) + E(X_2) + \cdots E(X_n)$ with or without independence.

$Var(X) = Var(X_1) + Var(X_2) + \cdots Var(X_n)$ with independence

$var(X) = \sum Var(X_i) + \sum\sum Covar(X_i, X_j)$ without independence.

used in Binomial, multinomial, Geometric, Hyper geometric

where ever; we can have 1 trial consisting of 'n' events at a time..

(iii) from MGF :-

$E(X)$ ✓

$var(X) = E(X^2) - (E(X))^2$ ✓ ✓

Eg: n people have hats, in a party; group all hats & take one each.

Y = no. of people who got their own hat.

Find $E(Y)$ & $var(Y)$.

Sol) $f_Y(y) = \dfrac{{}^nC_y \left((n-y)! - {}^{n-y}C_1 \cdot (n-y-1)! + {}^{n-y}C_2 (n-y-2)! \cdots\right)}{n!}$

derangements.... Soodhi formula.

can't use this for $E(y)$.

now; $Y = X_1 + X_2 + \cdots X_n$; here each (X_i, X_j) is not independent.

where X_i is bernoulli with $p = 1/n$

$\therefore E(Y) = \sum E(X_i) = 1$.

$var(Y) = \sum var(X_i) + \sum covar(X_i, X_j)$

$= n \cdot \frac{1}{n}(1 - \frac{1}{n}) + n(n-1)\left(covar(X_1, X_2)\right)$

$= E(X_1 X_2) - E(X_1) \cdot E(X_2)$

$= \frac{1}{n} \times \frac{1}{n-1} - \frac{1}{(n)^2}$

$= \frac{1}{n^2(n-1)}$

$= 1$

PARAMETER ESTIMATION:-

* let $x_1, x_2, \ldots x_n$ be a random sample from a distribution F_θ, where we don't know the parameters θ. (But we do know the family of F_θ)

Eg: • poission; with unknown parameter λ. ", we need to estimate this; with knowledge of $x_1, x_2, \ldots x_n$.

• normal dist; with unknown mean μ, σ^2.

* In probability theory; we take that the parameters are known; whereas in Statistics theory; the opp. is true; we use observed data to inference on parameter

→ Maximum Likelihood Estimators:- (of a parameter)
ML estimator.

* Also called 'point estimate'; since we give a single value for θ, instead of a range. (confidence interval) more Statistics like.

Note: Any statistic; used to estimate the value of a parameter is called estimator. There are many kinds of estimators

- observed value of estimator is estimate.

$\theta = 2$; irrespective of sample is an estimator (very bad one).

* Say; our sample is $x_1, x_2, x_3 \ldots x_n$. then;

$$f(x_1, x_2, \ldots x_n) = f(x_1) \cdot f(x_2) \ldots f(x_n)$$ x_i, x_j all are independent.

now; i) we know the distribution; but not the parameter θ.

ii). we choose a $\theta = \bar{\theta}$ such that the chances of $x_1, x_2, \ldots x_n$ is maximum!

iii) i.e. we find $\theta = \bar{\theta}$ such that $f(x_1, x_2, \ldots x_n)$ is maximum.

$\therefore$ $x_1, x_2, \ldots x_n$ is "maximu likely"

Hence $\boxed{\text{MLE of } \theta}$

→ lets calculate parameters for some distribution; by MLE :-

1) Bernoullie:-

say samples were $x_1, x_2, x_3, \ldots x_n$ ($\forall i, x_i \in \{0,1\}$)

let p be the parameter of the bernoullie dist.

then $f(x_1) = p^{x_1}(1-p)^{1-x_1}$ true for $x_1 = 0$ or $x_1 = 1$

$\therefore f(x_1, x_2, \ldots x_n) = p^{x_1+x_2+\cdots} \cdot (1-p)^{n-x_1-x_2-\cdots}$

$$\boxed{\begin{array}{l}\text{maximize } f(x_1, x_2 \ldots x_n) \\ \therefore \text{maximize } \log(f(x_1, x_2, \ldots x_n))\end{array}}$$ important na.

$\therefore \log(f(x_1, x_2 \ldots x_n)) = (x_1 + x_2 + \cdots x_n)\log p + (n - x_1 - x_2 - \cdots)\log(1-p)$

$\frac{\partial}{\partial p} = 0$

$\Rightarrow \frac{1}{p}(\Sigma x_i) - \frac{(n - \Sigma x_i)}{1-p} = 0$

$\therefore (1-p)(\Sigma x_i) = pn - p\Sigma x_i$

$\therefore \boxed{p = \frac{\Sigma x_i}{n}}$

this is MLE for p.

2) Poission:-

$f(x_i) = \frac{e^{-\lambda} \lambda^i}{i!}$

$\therefore f(x_1, x_2 \ldots x_n) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$

$\log(f(x_1 \text{ to } x_n)) = n \cdot \log(e^{-\lambda}) + (\Sigma x_i) \cdot \log \lambda - \{\text{const.}\}$

$\frac{\partial}{\partial \lambda} = 0$

$\Rightarrow 0 = n(-1) + \frac{\Sigma x_i}{\lambda}$

$\therefore \boxed{\lambda = \frac{\Sigma x_i}{n}}$

MLE of λ.

3) MLE for gaussian:-

consider $$\mathcal{N}(\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\therefore \quad f(x_1 \text{ to } x_n) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\therefore \quad \log(f(x_1 \text{ to } x_n)) = \sum_{i=1}^{n} -\frac{(x_i-\mu)^2}{2\sigma^2} - n\log(\sigma) + \{\text{const}\}$$

• now; we hav to maximize wrt 2 variables.

1) $$\frac{\partial \log f}{\partial \mu} = 0$$

$$\Sigma(x_i - \mu) = 0$$

$$\therefore \quad \underbrace{\boxed{\bar{\mu} = \frac{\Sigma x_i}{n}}}_{\text{MLE for } \mu.}$$

2) $$\frac{\partial \log f}{\partial \sigma} = 0$$

$$\Sigma -\frac{(x_i-\mu)^2}{2} \frac{(-2)}{\sigma^3} - \frac{n}{\sigma} = 0$$

$$\therefore \quad \underbrace{\boxed{\bar{\sigma}^2 = \frac{\Sigma (x_i-\mu)^2}{n}}}_{\text{MLE for } \sigma^2}$$

we see that:-

MLE(mean) = mean($x_1 x_2 \ldots x_n$)

MLE(var) = var($x_1 x_2 \ldots x_n$)

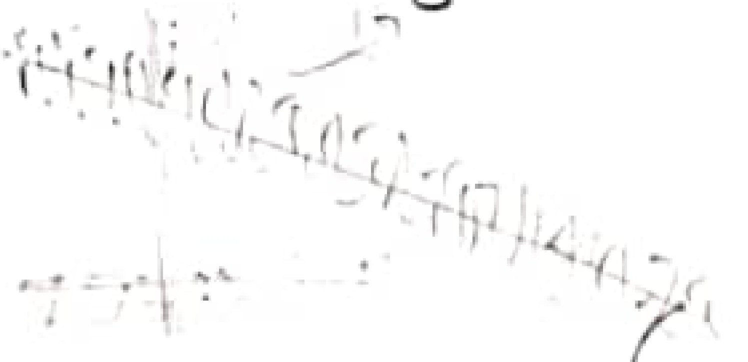
→ for other distribution to!
Bernoulie,
Poission.

actual solving doesnt involve seeing max. likelyhood at all!

→ ML for Least Square Line fitting:-

Linear regression:-

gaussian ε_i



$y = mx + c$

* values are pairs (x_i, y_i)

our distribution is:

• not a probability distribution.

$$\boxed{y_i = mx_i + c + \varepsilon_i \quad \text{from } \mathcal{N}(0, \sigma^2)}$$

• we know x_i - accurately.
• we have noisy y_i.

we need to determine m, c

even though ε_i are iid; y_i are only independent

$$y_i \in \mathcal{N}(mx_i + c, \sigma^2)$$

$$\therefore \quad P(y_i ; x_i, m, c) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(y_i - mx_i - c)^2}{2\sigma^2}}$$

σ is given?

$$\therefore \prod_{i=1}^{n} (P(y_i, x_i, m, c)) = \prod_{i=1}^{n} \frac{e^{-\frac{(y_i - mx_i - c)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$\therefore \log(f) = \sum_{i=1}^{n} -\frac{(y_i - mx_i - c)^2}{2\sigma^2} + \{\text{constant}\} \text{ in } \sigma.$$

* we need MLE for m, c:- (is σ given?)

$$\frac{\partial}{\partial m} = 0 \Rightarrow \sum x_i(y_i - mx_i - c) = 0$$

$$\Rightarrow \boxed{m(\sum x_i^2) + c(\sum x_i) = \sum x_i y_i}$$

$$\frac{\partial}{\partial c} = 0 \Rightarrow \sum(y_i - mx_i - c) = 0$$

$$\boxed{c = \left(\frac{\sum y_i}{n}\right) - m\left(\frac{\sum x_i}{n}\right)}$$

solve simultaneously

$$\therefore \boxed{\bar{m} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \qquad \boxed{\bar{c} = \bar{y} - \bar{m}\bar{x}}$$

where $\bar{x} = \frac{\sum x_i}{n}$

$\bar{y} = \frac{\sum y_i}{n}$

* indicator function:-

$I(X_i \le x_0)$ is a bernoullie rand. var; with parameter = $CDF(x_0)$

$I(X_i \in B_i)$ is a bernoullie;

where $B_i = [x_1, x_2]$ with $p = CDF(X_2) - CDF(X_1)$

Eg ∴ if $X = \{X_1, X_2, X_3, \ldots\}$

Eg. $N_{x_0} = \sum_{i=1}^{n} I(X_i \le X_0)$

then N_{x_0} is a binomial r.v.

* NOW; ML estimators are Random variables.

why? Bcoz it is a "function" of samples from an underlying dist.

poor choice).

- the ML estimator has its own pdf
  mean
  variance.

(When you calculate the value of MLE for a dataset this is a sample from pdf(MLE))

→ Bias; variance; mean square deviation of an estimator:- not just MLE.

$X_1, X_2, \ldots X_n$ are r.v. (iid) from a distribution with parameter θ.

- let $\hat{\theta}$ be an estimator of θ.

(How to decide, whether good or bad estimator?)

evaluate $(\hat{\theta}-\theta)^2$ (square deviation)

But it is R.V.

So; evaluate $E((\hat{\theta}-\theta)^2)$

* $E[(\hat{\theta}-\underbrace{\theta}_{\text{true value!}})^2]$ is called mean squared error of estimator.

(we desire low MSE estimators).

* if $E[\hat{\theta}]=\theta$, unbiased

  else, biased.

$(E(\hat{\theta})-\theta) \longrightarrow$ Bias of the estimator.

Eg: unbiased:-

- MLE of mean for a gaussian sample.
- MLE of variance for a gaussian sample (when mean is known).

biased:

- MLE of variance for a gaussian sample. (when mean is unknown)

★ interval of uniform dist.

$\hat{\theta} = \max(X_1, X_2, X_3 \ldots)$

$E(\hat{\theta}) = \int n x^{n-1}/\theta^n = n/n+1 \cdot \theta.$

* variance of estimator = $E[(\hat{\Theta} - E(\hat{\Theta}))^2]$.

need.... not the true mean of sample!

*

$$MSE(\hat{\Theta}) = \underbrace{E[(\hat{\Theta}-E(\hat{\Theta}))^2]}_{\text{variance}} + \underbrace{(E(\hat{\Theta})-\Theta)^2}_{\text{square bias}}$$

* A biased estimator may have lower MSE; owing to its low variance.

(so; saying unbiased better than biased is B.S.)

- also; if the MSE is not going down as the 'n' increases; then estimator is undesirable.

Eg: let $x_1, x_2, x_3 \cdots$ be r.v. of dist. of true parameter Θ.

if $\hat{\Theta} = x_1$

now; $E(\hat{\Theta}) = \Theta$

but $[Var(\hat{\Theta}) = \sigma^2$ (of dist)$]$

and! this dosen't go down! with n.

→ Estimator consistency:-

let Θ be the parameter of a dist.

& $\hat{\Theta}$ be a value of estimator of Θ.

- we say estimator is (asymptotically) consistent if

MSE → 0 as n → ∞.

probability

$$\lim_{n \to \infty} P(|\hat{\Theta}-\Theta| > \varepsilon) = 0 \quad \text{for any } \varepsilon > 0.$$

probability is zero...

like; $\Theta = 50$

& for 1 sample set $\hat{\Theta}$ turns out to be 5

then for 10^8 samplesets; $\hat{\Theta}$ turns out to be 50.

So; probability wise, fine.

Eg: for a distribution; with true mean θ;

Sample be $X_1, X_2, X_3, \dots X_n$

take two estimators; $\theta' = 1$ (irrespective of sample set)

$\theta'' = X_1$

then:

θ':

biased

variance = 0

MSE is high

inconsistent estimator

θ'':

unbiased ($E(\theta'') = \theta$)

variance is high

MSE is high

inconsistent estimator.

$\boxed{\text{an estimator can be unbiased \& still be inconsistent.}}$

weak law of large nos won't apply here; bcoz θ'' is not $\frac{X_1 + X_2 + X_3 + \cdots}{n}$ form.

→ Motivation for MLE:-

hard facts.

- MLE is a consistent estimator.

(as long as true values won't change with n; sample count)

- no consistent estimator; can achieve a lower asymptotic MSE than MLE.

Bias & variance of various ML Estimators:-

i) μ, σ^2 estimators for Gaussian:

$$\hat{\mu} = \frac{\Sigma x_i}{n}$$

- $E(\hat{\mu}) = \frac{n \cdot \mu}{n} = \mu$ unbiased
- $var(\hat{\mu}) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$ } variance decreasing with n.
- $MSE(\hat{\mu}) = \frac{\sigma^2}{n}$ ($\because$ bias $=0$).

$$\hat{\sigma^2} = \frac{\Sigma (x_i - \hat{\mu})^2}{N}$$

if μ is known beforehand:-

$$E(\hat{\sigma^2}) = \frac{1}{N} \Sigma E((x_i - \mu)^2)$$

$$= E((x_i - \mu)^2)$$

$$= \sigma^2$$

if μ not known:-

$$E(\hat{\sigma^2}) = \frac{1}{N} \cdot \Sigma E((x_i - \hat{\mu})^2)$$

$$= \frac{1}{n}\left[\Sigma E(x_i^2) + \Sigma E(\hat{\mu}^2) - 2\Sigma E(x_i \hat{\mu}) \right]$$

$$\boxed{E(\hat{\sigma^2}) = \sigma^2\left(1 - \frac{1}{n}\right)}$$

Biased.

But; bias ↓ as n↑.

2) θ estimator; for Uniform $[0, \theta]$:-

$$\hat{\theta} = \max(x_1, x_2, ---)$$

what is the distribution of $\hat{\theta}$?

$$P(\hat{\theta} \le x) = P(\max(x_1, x_2 --) \le x)$$

$$= P(x_1 \le x) \cdot P(x_2 \le x) \cdot ....$$

$$= \frac{x}{\theta} \cdot \frac{x}{\theta} ---$$

$$P(\hat{\theta} \le x) = \frac{x^n}{\theta^n}$$

$$\therefore \boxed{f_{\hat{\theta}}(x) = \frac{n x^{n-1}}{\theta^n}} \longrightarrow \boxed{E(\hat{\theta}) = \frac{n}{n+1}\theta.}$$

for $x \in [0, \theta]$

$= 0$ otherwise.

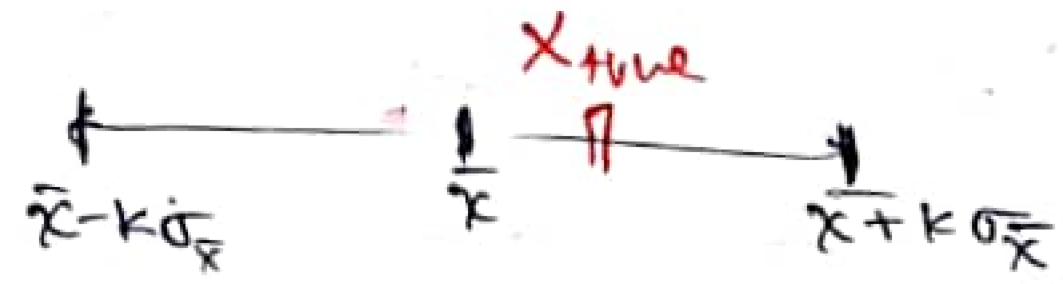
→ confidence intervals:-

→ θ_{true} might not be $\hat{\theta}$ in most of the cases. (point estimation).

So; we say;

$\theta_{true} \in [\hat{\theta} - c), \hat{\theta} + c)]$ with 99% probability.

confidence intervals.



Eg: 1) ML estimate of Gaussian mean:-

$x_1 x_2 \dots x_n$ are gaussian iids

$$\bar{X} = \frac{x_1 + x_2 + x_3 + \dots x_n}{n}$$

now;

$$\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0,1)$$] true enough; if X_i are not gaussian also. (CLT).

$$\therefore \quad P\left(-2.5 \le \sqrt{n}\left(\frac{\bar{x} - \mu}{\sigma}\right) \le 2.5\right) \simeq 0.99$$

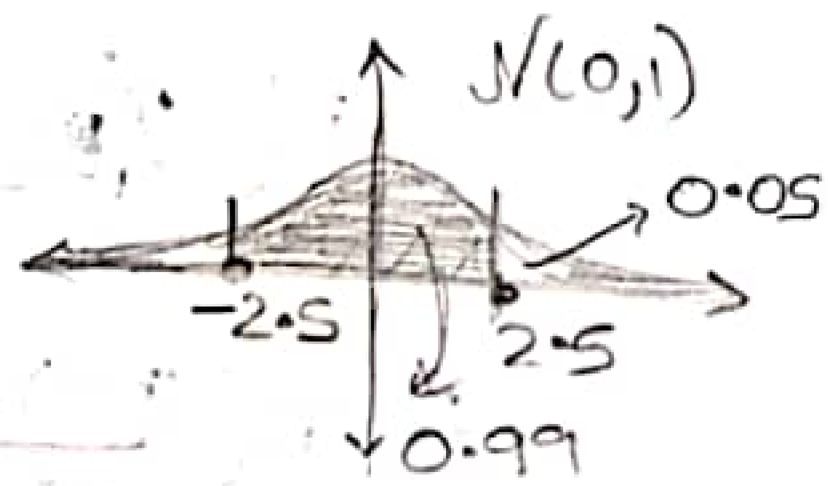
if we don't know σ_{true}; approximate with $\sigma_{dataset}$.

$$\therefore \quad P\left(\bar{X} - \frac{2.5\sigma}{\sqrt{n}} \le \mu \le \bar{x} + \frac{2.5\sigma}{\sqrt{n}}\right) \simeq 0.99$$

$\sigma_{\bar{x}}$

$\therefore \mu$ (true value) lies in $\left[\bar{x} - \frac{2.5\sigma}{\sqrt{n}}, \bar{x} + \frac{2.5\sigma}{\sqrt{n}}\right]$

with 99% confidence.



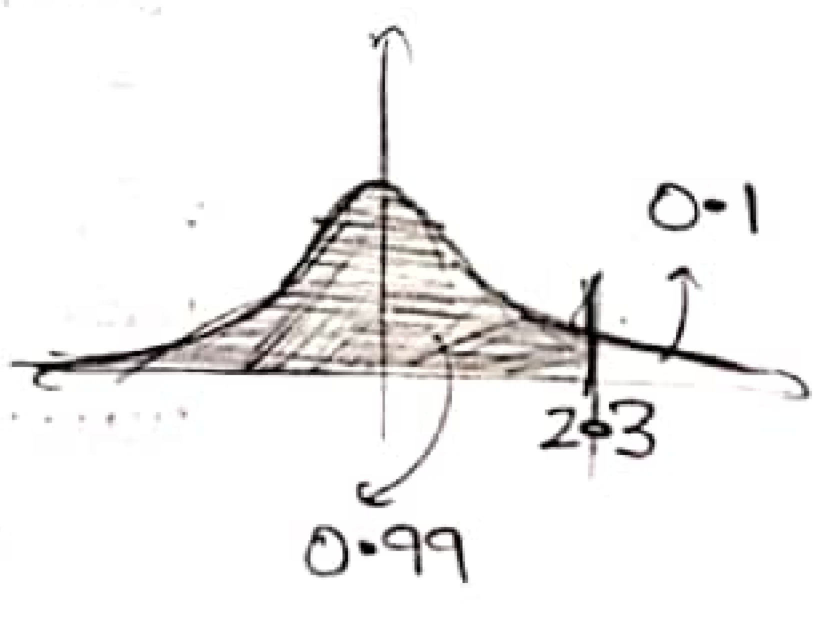
also;

$$P\left(\sqrt{n}\left(\frac{\bar{x} - \mu}{\sigma}\right) \le 2.3\right) \simeq 0.99$$

$$\therefore P\left(\bar{x} - \frac{2.3\sigma}{\sqrt{n}} \le \mu\right) \simeq 0.99.$$

one sided confidence...

μ lies right of ( ) with 99% confidence.



$\therefore Z_{0.05} = 2.5$

$Z_{0.1} = 2.3$ for $N(0,1)$

→ 2. for Variance's MLEstimator:-

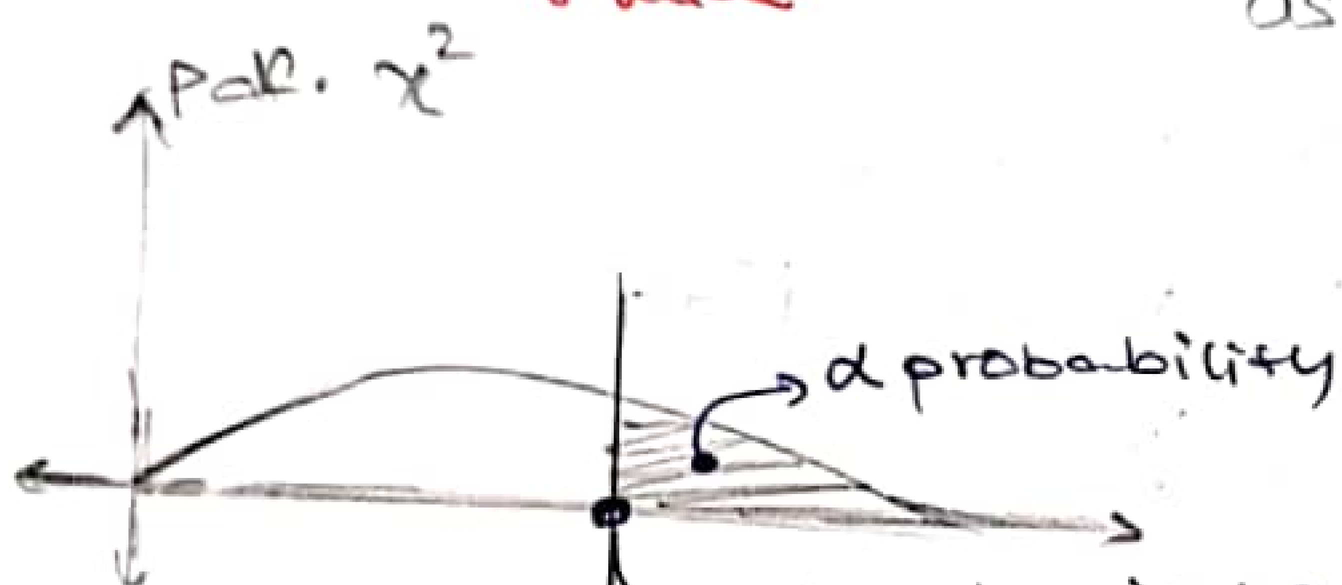
$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$$

& as we've already seen;

$$(n-1)\left(\frac{s^2}{\sigma^2}\right) \sim \chi^2_{n-1}$$

chi square dist^n.
(also gaussian as n increases).

true σ value.



this value is termed Z_α.
or
$\chi^2_{\alpha,n-1}$.

$\boxed{P(X \geq Z_\alpha) = \alpha}$

∴ now;

$$P\left((n-1)\frac{s^2}{\sigma^2} \geq \chi^2_{\alpha/2,n-1}\right) \approx \frac{\alpha}{2}$$

$$P\left((n-1)\frac{s^2}{\sigma^2} \geq \chi^2_{1-\frac{\alpha}{2},n-1}\right) \approx 1-\frac{\alpha}{2}$$

$$\therefore P\left(\chi^2_{1-\frac{\alpha}{2},n-1} \leq \frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{\frac{\alpha}{2},n-1}\right) \approx 1-\alpha.$$

$$\therefore \boxed{P\left(\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right) \approx 1-\alpha}$$

these values are usually avaliable in tabular manner.

confidence interval.